Inverse Problems in Astrophysics

•Part 1: Introduction inverse problems and image deconvolution

•Part 2: Introduction to Sparsity and Compressed Sensing

•Part 3: Wavelets in Astronomy: from orthogonal wavelets and to the Starlet transform.

•Part 4: Beyond Wavelets

•Part 5: Inverse problems and their solution using sparsity: denoising, deconvolution, inpainting, blind source separation.

•Part 6: CMB & Sparsity

•Part 7: Perspective of Sparsity & Compressed Sensing in Astrophsyics

Data Representation Tour

- Computational harmonic analysis seeks representations of a signal as linear combinations of basis, frame, dictionary, element :

$$s_i = \sum_{k=1}^{K} \underbrace{\alpha_k}_{\text{coefficients}} \underbrace{\phi_k}_{\text{basis, frame}}$$

- Fast calculation of the coefficients α_k
- Analyze the signal through the statistical properties of the coefficients

What is a good sparse representation for data?

A signal *s* (*n* samples) can be represented as sum of weighted elements of a given dictionary

Dictionary (basis, frame)

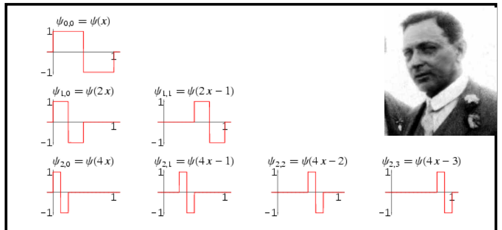
$$\Phi = \{\phi_1, \ldots, \phi_K\}$$

Atoms

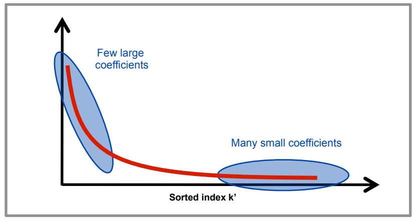
$$s = \sum_{k=1}^{K} \alpha_k \phi_k = \Phi\alpha$$

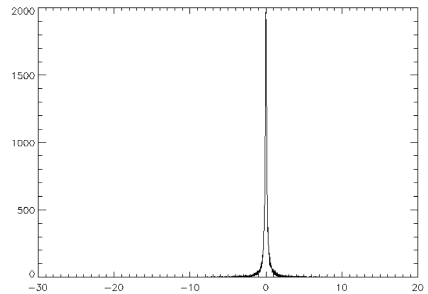
coefficients

Ex: Haar wavelet



$$f(x) = c_0 + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} c_{jk}\psi_{jk}(x).$$





- Fast calculation of the coefficients
- Analyze the signal through the statistical properties of the coefficients
- Approximation theory uses the sparsity of the coefficients

The Great Father Fourier - Fourier Transforms

Any Periodic function can be expressed as linear combination of basic trigonometric functions

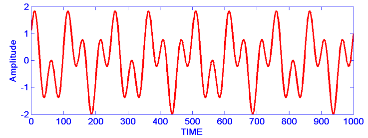
(Basis functions used are sine and cosine)

Jean-Baptiste-Joseph Fourier
(1768-1830)

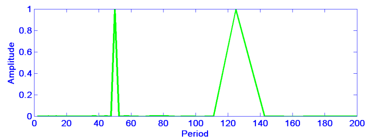
$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-2\pi i f t} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{2\pi i f t} df$$

Time domain



Frequency domain

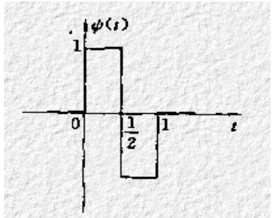


Alfred Haar Wavelet (1909):

The first mention of wavelets appeared in an appendix to the thesis of Haar

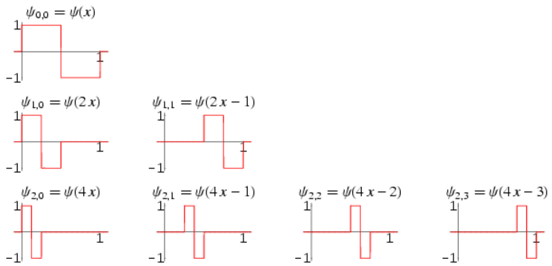
- **With *compact support,* vanishes outside of a finite interval**
- **Not continuously differentiable**
- **Wavelets are functions defined over a finite interval and having an average value of zero.**

$$f(x) = c_0 + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} c_{jk} \psi_{jk}(x).$$



$$\Psi(x) = \begin{cases} 1 & 0 \le x \le \frac{1}{2} \\ -1 & \frac{1}{2} < x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Haar wavelet



==> What kind of $\psi(t)$ could be useful?

. Impulse Function (Haar): Best time resolution

. Sinusoids (Fourier): Best frequency resolution

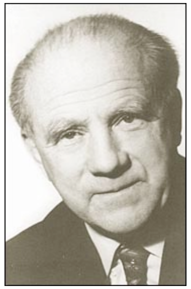
==> We want both of the best resolutions

==> Heisenberg, 1930

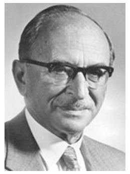
Uncertainty Principle

There is a lower bound for

$$\Delta t \cdot \Delta\omega$$



SFORT TIME FOURIER TRANSFORM (STFT)

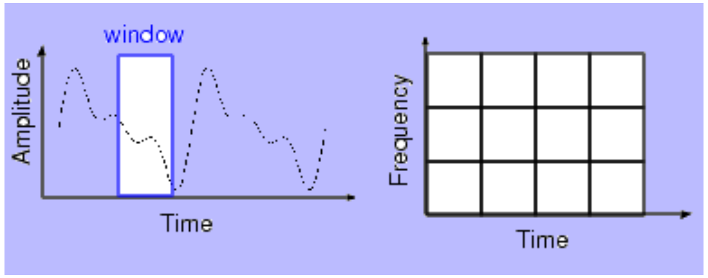


- **Dennis Gabor (1946) Used STF**
To analyze only a small section of the signal at a time -- a technique called *Windowing the Signal*.
- **The Segment of Signal is Assumed *Stationary***

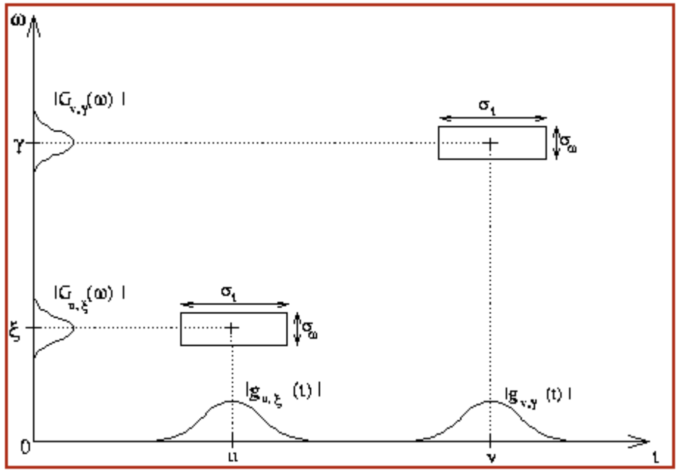
The Short Term Fourier Transform is defined by:

$$STFT(\nu, b) = \int_{\infty}^{+\infty} \exp(-j2\pi\nu t) f(t) g(t-b) dt$$

when g is a Gaussien, it corresponds to the Gabor transform.



Heisenberg Box

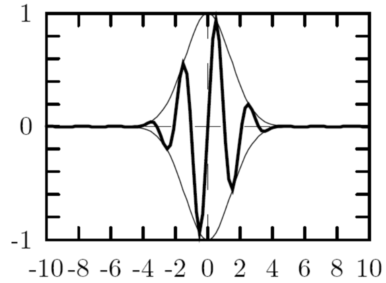


$$\boxed{\sigma_t^2\,\sigma_\omega^2 \geq \frac{1}{4}.}$$

Candidate analyzing functions for piecewise smooth signals

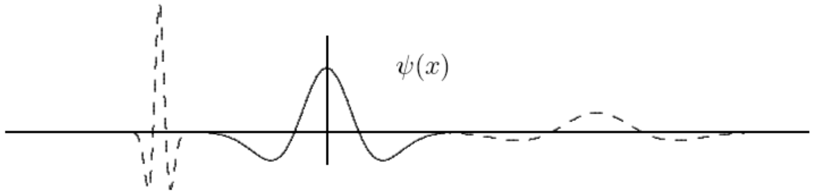
- Windowed fourier transform or Gaboriets :

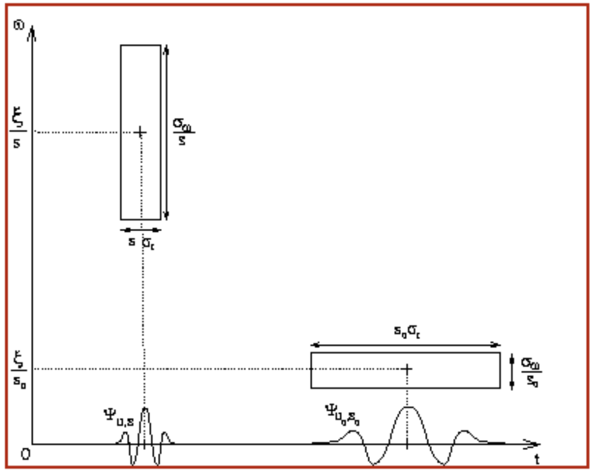
$$\psi_{\omega,b}(t) = g(t-b)e^{i\omega t}$$



- Wavelets :

$$\psi_{a,b} = \frac{1}{\sqrt{a}}\psi(\frac{t-b}{a})$$




ω
ξ/s
σ_ω/s
s σ_t
s_0σ_t
ξ/s_0
σ_ω/s_0
Ψ_u,s
Ψ_u0,s0
0
t

The Continuous Wavelet Transform

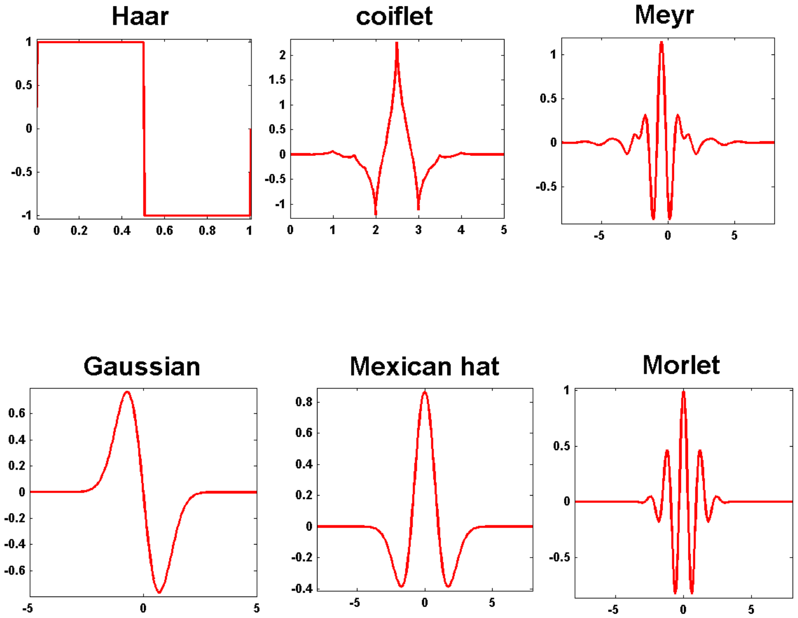
$$W(a,b) = K \int_{-\infty}^{+\infty} \psi^*(\frac{x-b}{a}) f(x) dx$$

where:

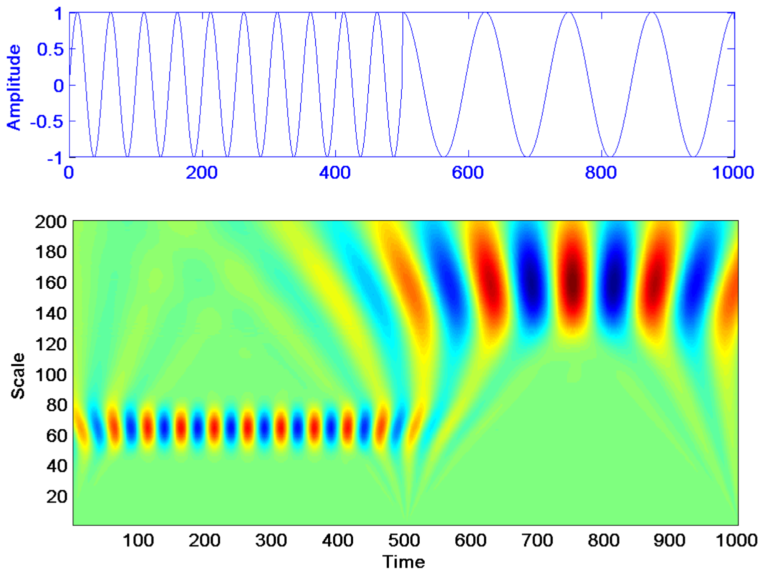
- $W(a,b)$ is the wavelet coefficient of the function $f(x)$
- $\psi(x)$ is the analyzing wavelet
- a (> 0) is the scale parameter
- b is the position parameter

In Fourier space, we have: $\hat{W}(a,\nu) = \sqrt{a}\hat{f}(\nu)\hat{\psi}^*(a\nu)$
When the scale a varies, the filter $\hat{\psi}^*(a\nu)$ is only reduced or dilated while keeping the same pattern.

Some typical mother wavelets



Typical picture



The Inverse Transform

The inverse transform is:

$$f(x) = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \frac{1}{\sqrt{a}} W(a,b) \psi(\frac{x-b}{a}) \frac{dadb}{a^2}$$

where

$$C_\psi = \int_{-\infty}^{+\infty} \mid \hat{\psi}(t) \mid^2 \frac{dt}{t} < +\infty$$

Reconstruction is only possible if C_ψ is defined (admissibility condition).
This condition implies $\hat{\psi}(0) = 0$, i.e. the mean of the wavelet function is 0.

Yves Meyer

A Major Breakthrough

Daubechies, 1988 and Mallat, 1989

Daubechies:

Compactly Supported Orthogonal and Bi-Orthogonal Wavelets

Mallat:

Theory of Multiresolution Signal Decomposition

Fast Algorithm for the Computation of Wavelet Transform Coefficients using Filter Banks

Multiresolution Analysis

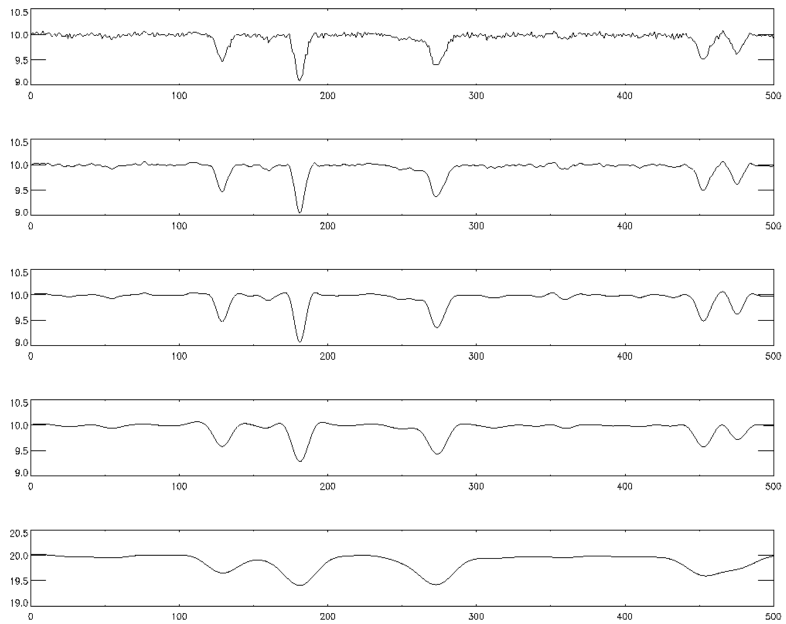
The multiresolution analysis (Mallat, 1989) results from the embedded subsets generated by the interpolations at different scales.

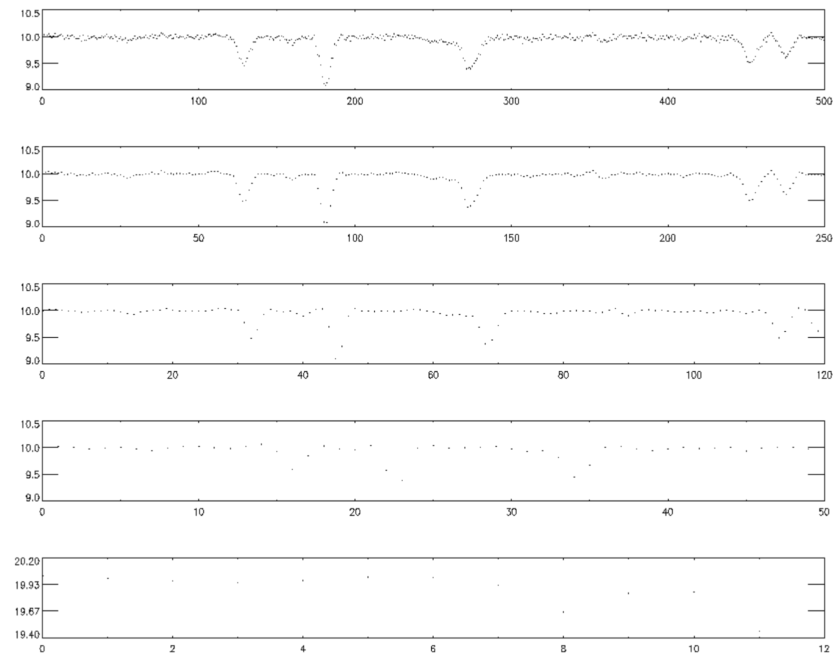
A function $f(x)$ is projected at each step j on the subset V_j $(\ldots\ldots \subset V_3 \subset V_2 \subset V_1 \subset V_0)$. This projection is defined by the scalar product $c_{j,k}$ of $f(x)$ with the scaling function $\phi(x)$ which is dilated and translated:

$$c_{j,k} \quad = \quad < f(x), \phi_{j,k}(x) >$$

$$\phi_{j,k}(x) = 2^{-j}\phi(2^{-j}x - k)$$

where $\phi(x)$ is the scaling function. ϕ is a low-pass filter.


10.5
10.0
9.5
9.0
0
100
200
300
400
500
20.5
20.0
19.5
19.0



10.5
10.0
9.5
9.0
0
100
200
300
400
500
50
150
250
20
40
60
80
120
10
30
20.20
19.93
19.67
19.40
2
4
6
8
12

Wavelets and Multiresolution Analysis

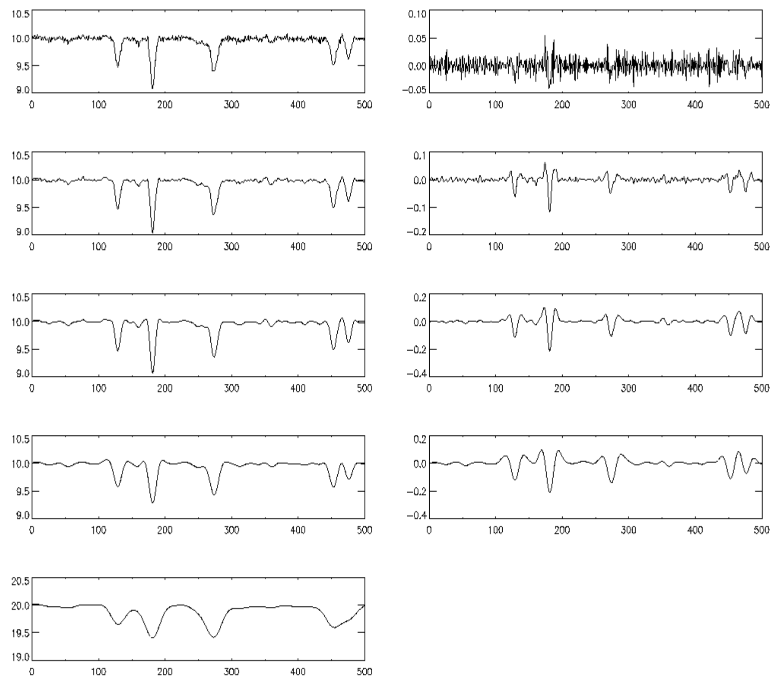
The difference between c_j and c_{j+1} is contained in the detail signal belonging to the space O_{j+1} orthogonal to V_{j+1}.

$$O_{j+1} \oplus V_{j+1} = V_j$$

The set $\{\sqrt{2^{-j}}\psi(2^{-j}x - k)\}_{k \in \mathcal{Z}}$ form a basis of O_j. $\psi(x)$ is the wavelet function.

the wavelet coefficients are obtained by:

$$w_{j,k} \quad = \quad < f(x), 2^{-j}\psi(2^{-j}x - k) >$$



10.5
10.0
9.5
9.0
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100
200
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500
0.10
0.05
0.00
-0.05
0.1
0.0
-0.1
-0.2
0.2
-0.4
20.5
20.0
19.5
19.0

the Fast Wavelet Transform

As $\phi(x)$ is a scaling function which has the property: $\frac{1}{2}\phi(\frac{x}{2}) = \sum_n h(n)\phi(x-n)$. $c_{j+1,k}$ can be obtained by direct computation from $c_{j,k}$

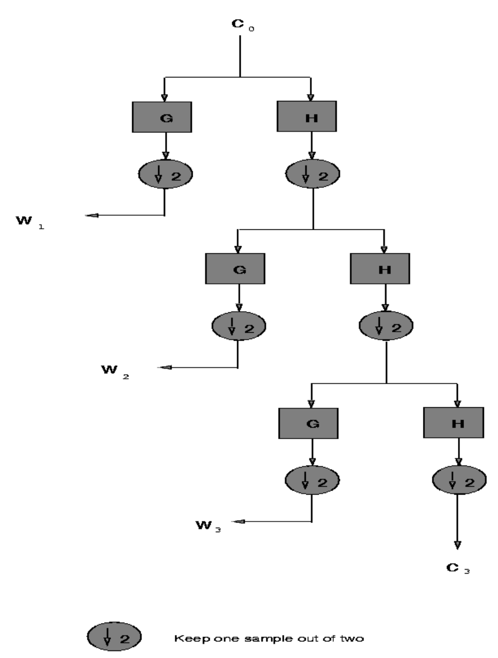
$$c_{j+1,k} = \sum_n h(n-2k)c_{j,n}$$

and $\frac{1}{2}\psi(\frac{x}{2}) = \sum_n g(n)\phi(x-n)$.
The scalar products $< f(x), 2^{-(j+1)}\psi(2^{-(j+1)}x-k) >$ are computed with:

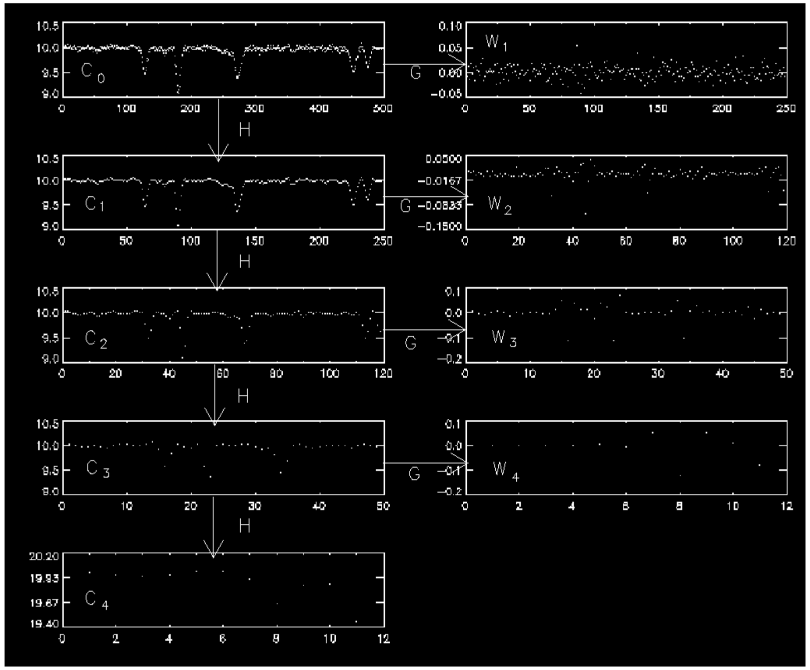
$$w_{j+1,k} = \sum_n g(n-2k)c_{j,n}$$

Reconstuction by:

$$c_{j,k} = 2\sum_n h(k-2n)c_{j+1,n} + g(k-2n)w_{j+1,n}$$



c_0
G
H
$\downarrow 2$
$\downarrow 2$
w_1
G
H
$\downarrow 2$
$\downarrow 2$
w_2
G
H
$\downarrow 2$
$\downarrow 2$
w_3
c_3
$\downarrow 2$
Keep one sample out of two



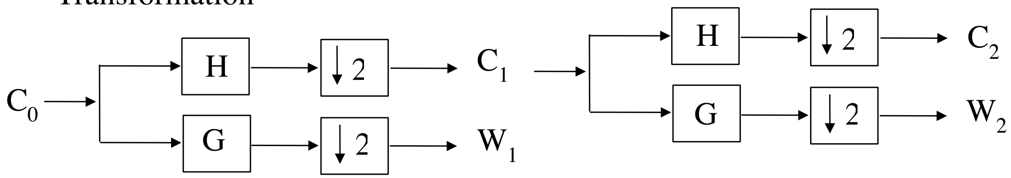
C 0
W 1
C 1
W 2
C 2
W 3
C 3
W 4
C 4
G
H

64	48	16	32	56	56	48	24
56	24	56	36	**8**	**-8**	**0**	**12**
40	46	**16**	**10**	**8**	**-8**	**0**	**12**
43	**-3**	**16**	**10**	**8**	**-8**	**0**	**12**

The Orthogonal Wavelet Transform (OWT)

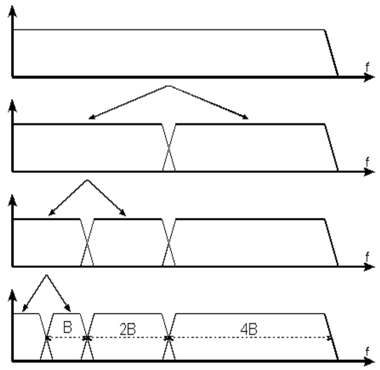
$$s_l = \sum_k c_{J,k}\phi_{J,l}(k) + \sum_k \sum_{j=1}^{J} \psi_{j,l}(k) w_{j,k}$$

Transformation



$$c_{j+1,l} = \sum_h h_{k-2l} c_{j,k} = (\bar{h} * c_j)_{2l}$$

$$w_{j+1,l} = \sum_h g_{k-2l} c_{j,k} = (\bar{g} * c_j)_{2l}$$



Reconstruction:

$$c_{j,l} = \sum_k \tilde{h}_{k+2l} c_{j+1,k} + \tilde{g}_{k+2l} w_{j+1,k} = \tilde{h} * \breve{c}_{j+1} + \tilde{g} * \breve{w}_{j+1}$$

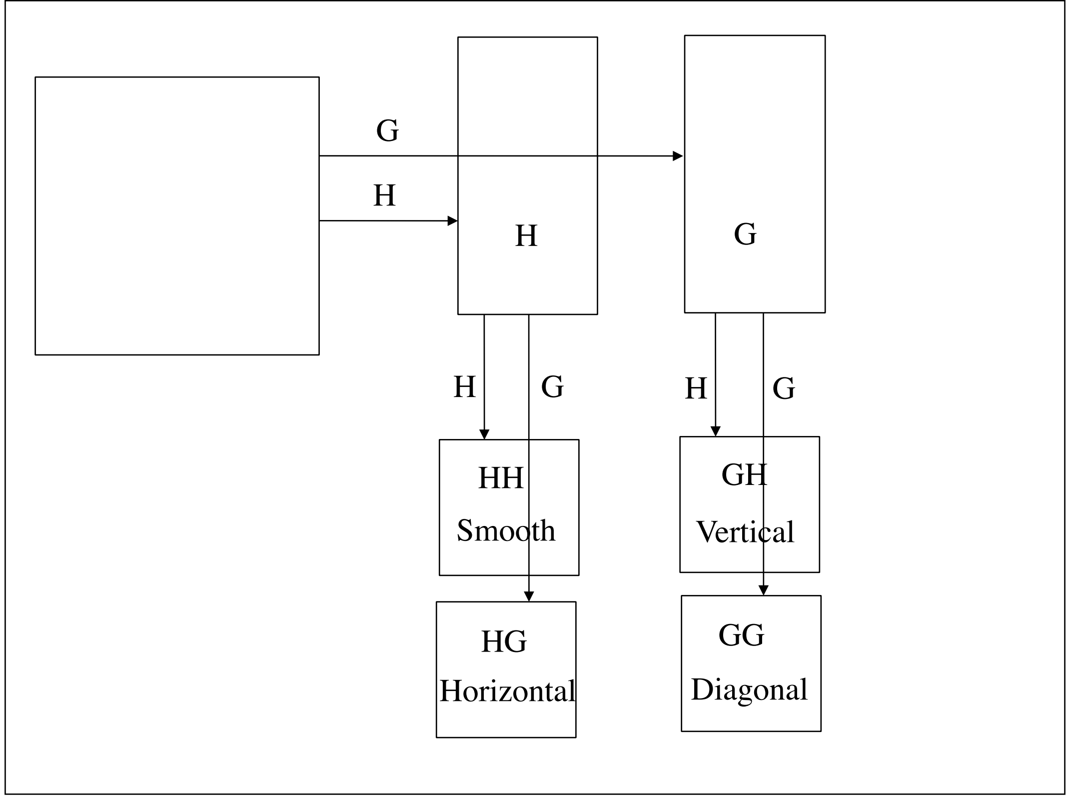
$$\breve{x} = (x_1, 0, x_2, 0, x_3, \ldots, 0, x_j, 0, \ldots, x_{n-1}, 0, x_n)$$

At two dimensions, we separate the variables x,y:

- vertical wavelet: $\psi^1(x,y) = \phi(x)\psi(y)$
- horizontal wavelet: $\psi^2(x,y) = \psi(x)\phi(y)$
- diagonal wavelet: $\psi^3(x,y) = \psi(x)\psi(y)$

The detail signal is contained in three sub-images

$$\begin{aligned}
w_j^1(k_x,k_y) &= \sum_{l_x=-\infty}^{+\infty}\sum_{l_y=-\infty}^{+\infty} g(l_x-2k_x)h(l_y-2k_y)c_{j+1}(l_x,l_y) \\
w_j^2(k_x,k_y) &= \sum_{l_x=-\infty}^{+\infty}\sum_{l_y=-\infty}^{+\infty} h(l_x-2k_x)g(l_y-2k_y)c_{j+1}(l_x,l_y) \\
w_j^3(k_x,k_y) &= \sum_{l_x=-\infty}^{+\infty}\sum_{l_y=-\infty}^{+\infty} g(l_x-2k_x)g(l_y-2k_y)c_{j+1}(l_x,l_y)
\end{aligned}$$


G
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Vertical
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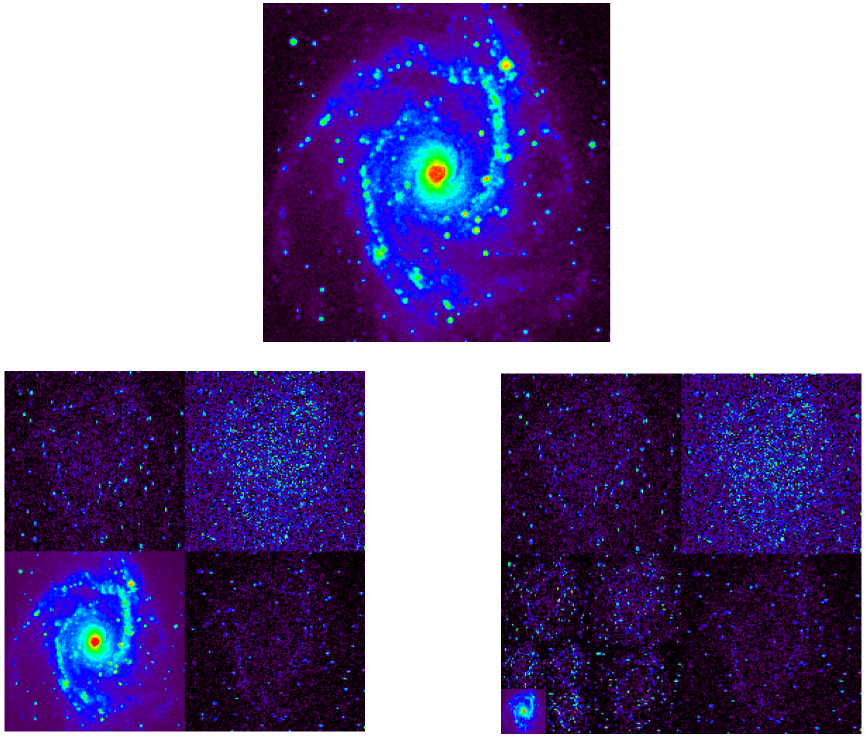
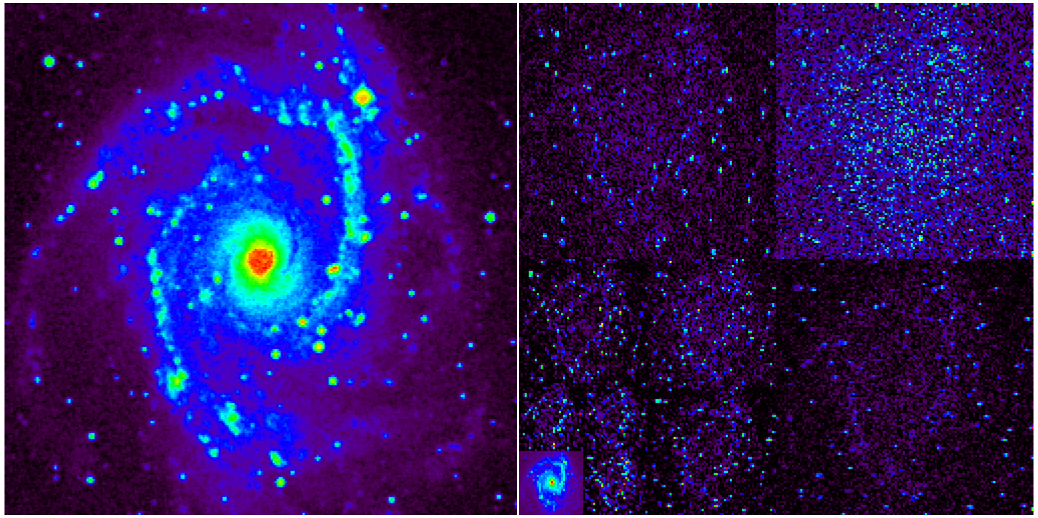
Diagonal



NGC2997 NGC2997 WT

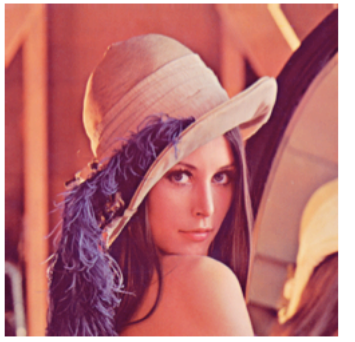


JPEG/JPEG 2000

Original BMP

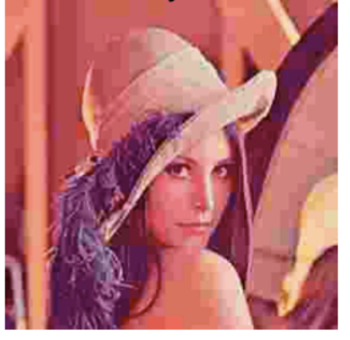
300x300x24

270056 bytes



JPEG 1:68

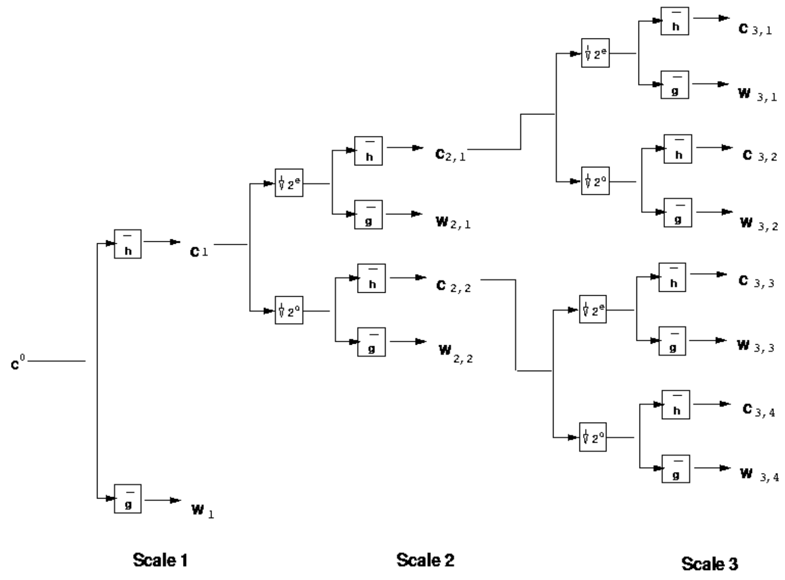
3983 bytes



JPEG2000 1:70

3876 bytes

1D undecimated wavelet transform



The à trous Algorithm

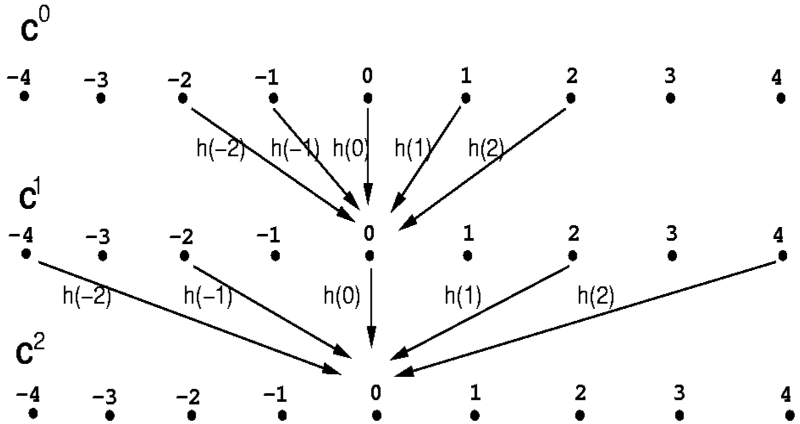
It exists however a very efficient way to implement it. The "à trous" algorithm consists in considering the filter $h^{(j)}$ instead of h where $h_l^{(j)} = h_l$ if $l/2^j$ is an integer and 0 otherwise. For example, we have $h^{(1)} = (\ldots, h_{-2}, 0, h_{-1}, 0, h_0, 0, h_1, 0, h_2, \ldots)$. Then $c_{j+1,l}$ and $w_{j+1,l}$ can be expressed by:

$$
\begin{aligned}
c_{j+1,l} &= (\bar{h}^{(j)} * c_j)_l = \sum_k h_k c_{j,l+2^j k} \\
w_{j+1,l} &= (\bar{g}^{(j)} * c_j)_l = \sum_k g_k c_{j,l+2^j k}
\end{aligned}
$$

The reconstruction is obtained by:

$$
c_j = \frac{1}{2}(\tilde{h}^{(j)} * c_{j+1} + \tilde{g}^{(j)} w_{j+1})
$$

Passage from c_0 to c_1, and from c_1 to c_2

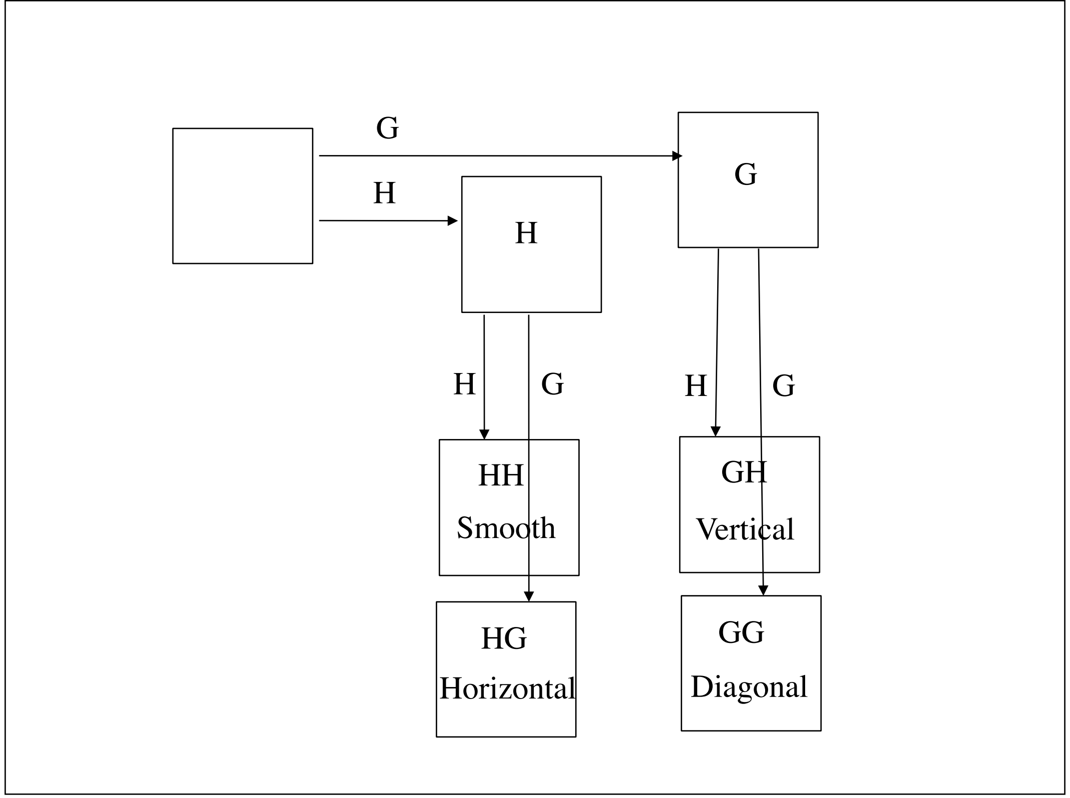


2D Undecimated Wavelet Transform

The à trous algorithm can be extended to 2D:

$$
\begin{aligned}
c_{j+1,k,l} &= (\bar{h}^{(j)}\bar{h}^{(j)} * c_j)_{k,l} \\
w_{j+1,1,k,l} &= (\bar{g}^{(j)}\bar{h}^{(j)} * c_j)_{k,l} \\
w_{j+1,2,k,l} &= (\bar{h}^{(j)}\bar{g}^{(j)} * c_j)_{k,l} \\
w_{j+1,3,k,l} &= (\bar{g}^{(j)}\bar{g}^{(j)} * c_j)_{k,l}
\end{aligned}
$$

where $hg * c$ is the convolution of c by the separable filter hg (i.e convolution first along the columns per h and then convolution along the lines per g).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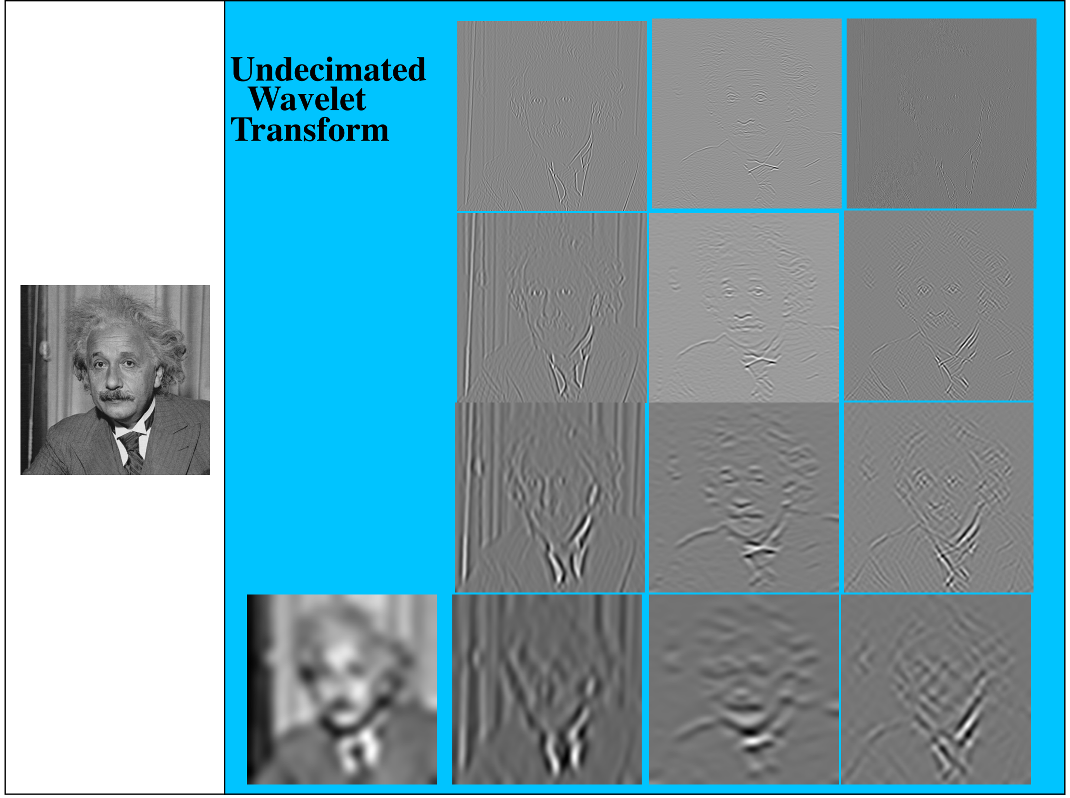
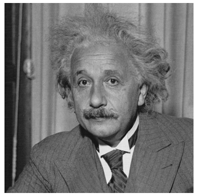
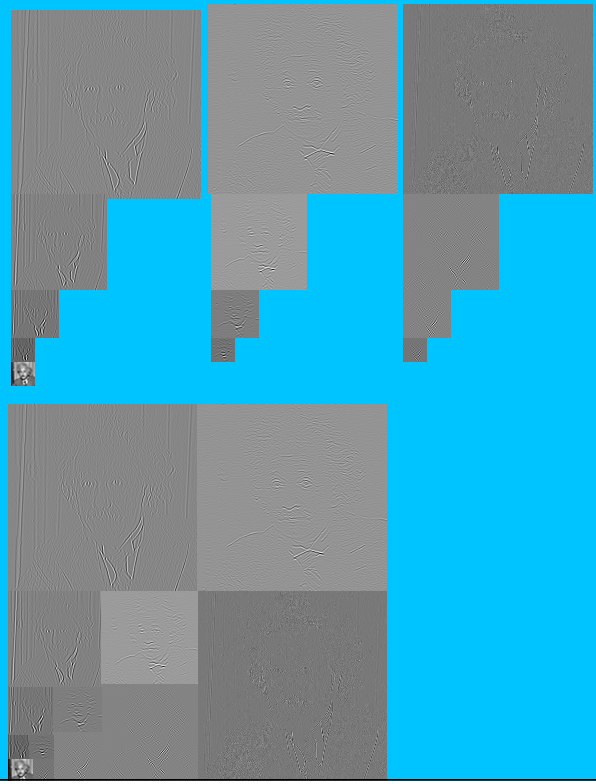
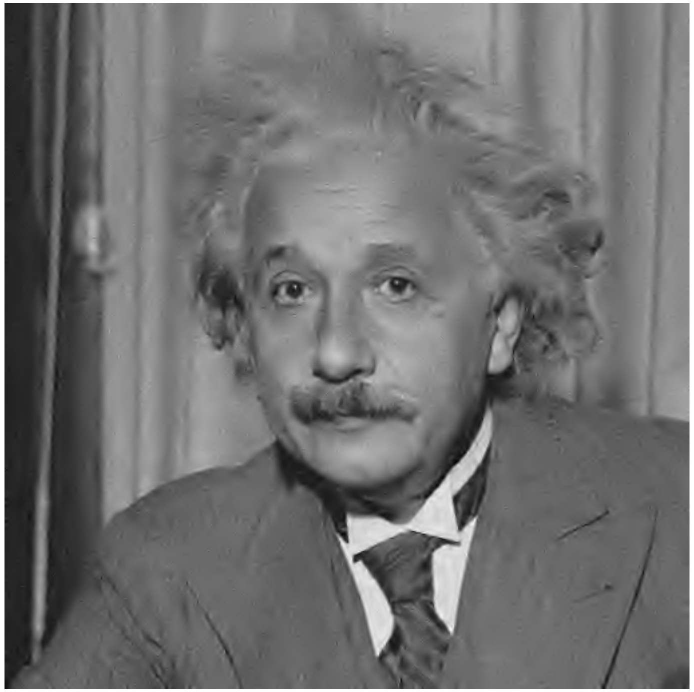

Undecimated
Wavelet
Transform



Partially Undecimated Wavelet Transform





Hard Threshold: 3sigma

OWT UWT

Redundancy	1	4	7	10	13
PSNR(dB)	28.90	30.58	31.51	31.83	31.89
Square Error	83.54	52.28	45.83	42.51	41.99

ISOTROPIC UNDECIMATED WT: The Starlet Transform

- Isotropic transform well adapted to astronomical images.
- Diadic Scales.
- Invariance per translation.

Scaling function and dilation equation:

$$\frac{1}{4}\varphi(\frac{x}{2},\frac{y}{2}) = \sum_{l,k} h(l,k)\varphi(x-l,y-k)$$

Wavelet function decomposition:

$$\frac{1}{4}\psi(\frac{x}{2},\frac{y}{2}) = \sum_{l,k} g(l,k)\varphi(x-l,y-k)$$

A trous wavelet transform:

$$w_j(x,y) = \langle f(x,y), \frac{1}{4^j}\varphi(\frac{x-l}{2^j},\frac{y-k}{2^j}) \rangle$$

Generally, the wavelet resulting from the difference between two successive approximations is applied:

$$w_{j+1,k} = c_{j,k} - c_{j+1,k}$$

The associated wavelet is $\psi(x)$.

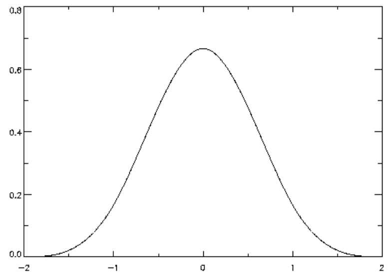
$$\frac{1}{2}\psi(\frac{x}{2}) = \phi(x) - \frac{1}{2}\phi(\frac{x}{2})$$

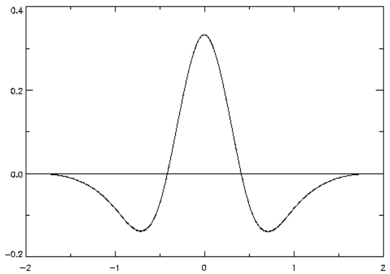
The reconstruction algorithm is immediate:

$$c_{0,k} = c_{J,k} + \sum_{j=1}^{J} w_{j,k}$$

The Isotropic Wavelet and Scaling Functions

$$
\begin{aligned}
B_3(x) &= \frac{1}{12}(\mid x-2\mid^3 -4\mid x-1\mid^3 +6\mid x\mid^3 -4\mid x+1\mid^3 + \mid x+2\mid^3) \\
\psi(x,y) &= B_3(x)B_3(y) \\
\frac{1}{4}\psi(\frac{x}{2},\frac{y}{2}) &= \phi(x,y) - \frac{1}{4}\phi(\frac{x}{2},\frac{y}{2})
\end{aligned}
$$





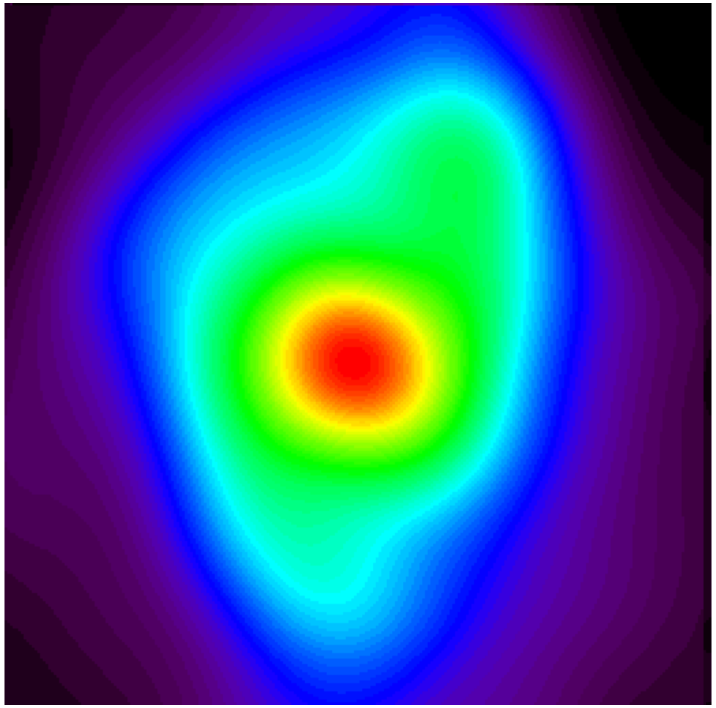
$$\begin{pmatrix} \frac{1}{16} & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} & \frac{1}{16} \end{pmatrix} \otimes \begin{pmatrix} 1/16 \\ 1/4 \\ 3/8 \\ 1/4 \\ 1/16 \end{pmatrix} = \begin{pmatrix} \frac{1}{256} & \frac{1}{64} & \frac{3}{128} & \frac{1}{64} & \frac{1}{256} \\ \frac{1}{64} & \frac{1}{16} & \frac{3}{32} & \frac{1}{16} & \frac{1}{64} \\ \frac{3}{128} & \frac{3}{32} & \frac{9}{64} & \frac{3}{32} & \frac{3}{128} \\ \frac{1}{64} & \frac{1}{16} & \frac{3}{32} & \frac{1}{16} & \frac{1}{64} \\ \frac{1}{256} & \frac{1}{64} & \frac{3}{128} & \frac{1}{64} & \frac{1}{256} \end{pmatrix}$$

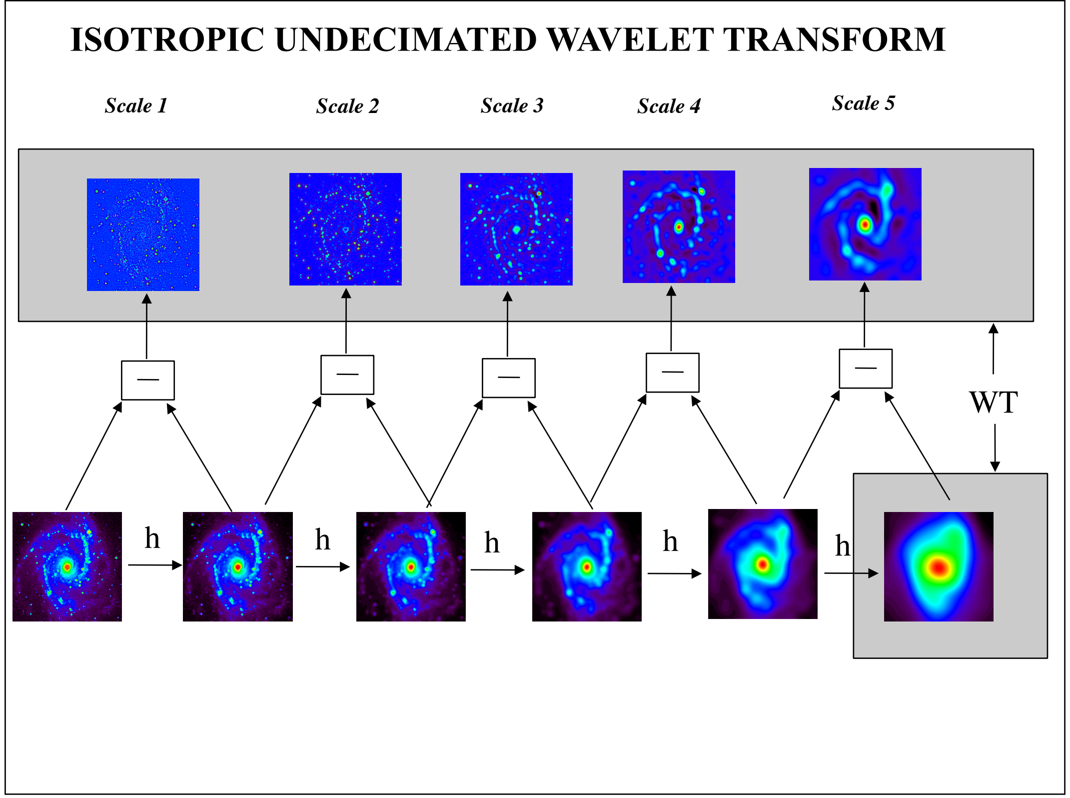
In the 2-dimensional case, we assume the separability, which leads to a row-by-row convolution with $(\frac{1}{16}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4}, \frac{1}{16})$; followed by column-by-column convolution.

Boundaries

The most general way to handle the boundaries is to consider that $c_{k+N} = c_{N-k}$ (mirror). But other methods can be used such as periodicity ($c_{k+N} = c_k$), or continuity ($c_{k+N} = c_N$).

NGC2997




ISOTROPIC UNDECIMATED WAVELET TRANSFORM
Scale 1
Scale 2
Scale 3
Scale 4
Scale 5
WT
h
h
h
h
h

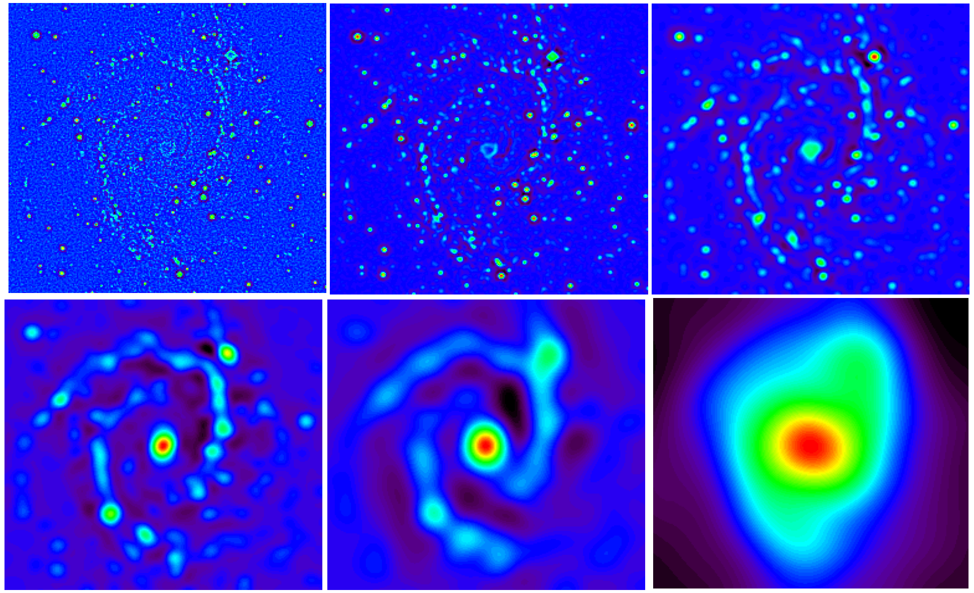
The STARLET Transform

Isotropic Undecimated Wavelet Transform (a trous algorithm)

$\varphi = B_3 - \text{spline}, \quad \frac{1}{2}\psi(\frac{x}{2}) = \frac{1}{2}\varphi(\frac{x}{2}) - \varphi(x)$

$h = [1,4,6,4,1]/16, \quad g = \delta - h, \quad \tilde{h} = \tilde{g} = \delta$

$$I(k,l) = c_{J,k,l} + \sum_{j=1}^{J} w_{j,k,l}$$

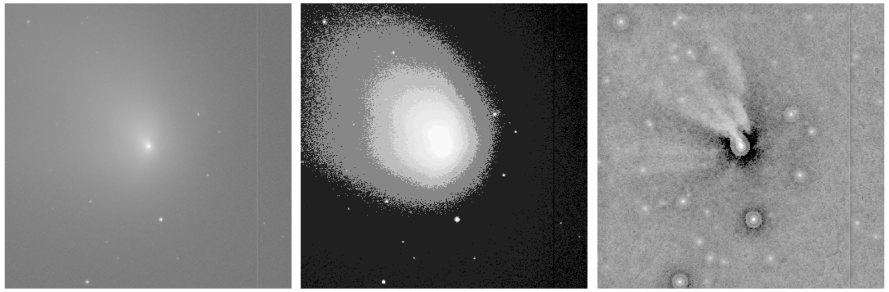


Dynamic Range Compression

Images with a high dynamic range are also difficult to analyze. For example, astronomers generally visualize their images using a logarithmic look-up-table conversion.

Wavelet can also be used to compress the dynamic range at all scales, and therefore allows us to clearly see some very faint features. For instance, the wavelet-log representations consists in replacing $w_{j,k,l}$ by $\log(|w_{j,k,l}|)$, leading to the alternative image

$$I_{k,l} = \log(c_{J,k,l}) + \sum_{j=1}^{J} \mathrm{sgn}(w_{j,k,l}) \log(\mid w_{j,k,l} \mid + \epsilon)$$



Left - Hale-Bopp Comet image. Middle - histogram equalization results, Right - wavelet-log representations.