Multiscale Transforms and Applications

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Status of the Wavelet Transform in Astronomy

The wavelet transform is used in many astronomical domains:

- galaxies counting: large scale structures analysis, void detection, ...
- Gamma: Gamma ray burst detection
- X-ray images (ROSAT, XMM, AXAF): extended sources and filaments detection
- Optics: asteroid and planetary ring detection, deconvolution
- Infrared (ISO project): calibration, source detection, deconvolution, ...
- Cosmic Microwave Background (Planck): fluctuation analysis
- Radio: aperture synthesis image deconvolution

and for all kind of astrophysics, from solar to CMB!

ADS Abstract Service

keyword in title: wavelet - multiscale - multiresolution
$\rightarrow$ more than 500 papers!

Ressources

– www.wavelet.org

– Time-Frequency: Time-Frequency Analysis, L. Cohen, Printice Hall Signal processing series, A.V. Oppenheim editor, 1995.

– Wavelet: a Wavelet Tour of signal processing, S. Mallat, Academic Press, 1998.

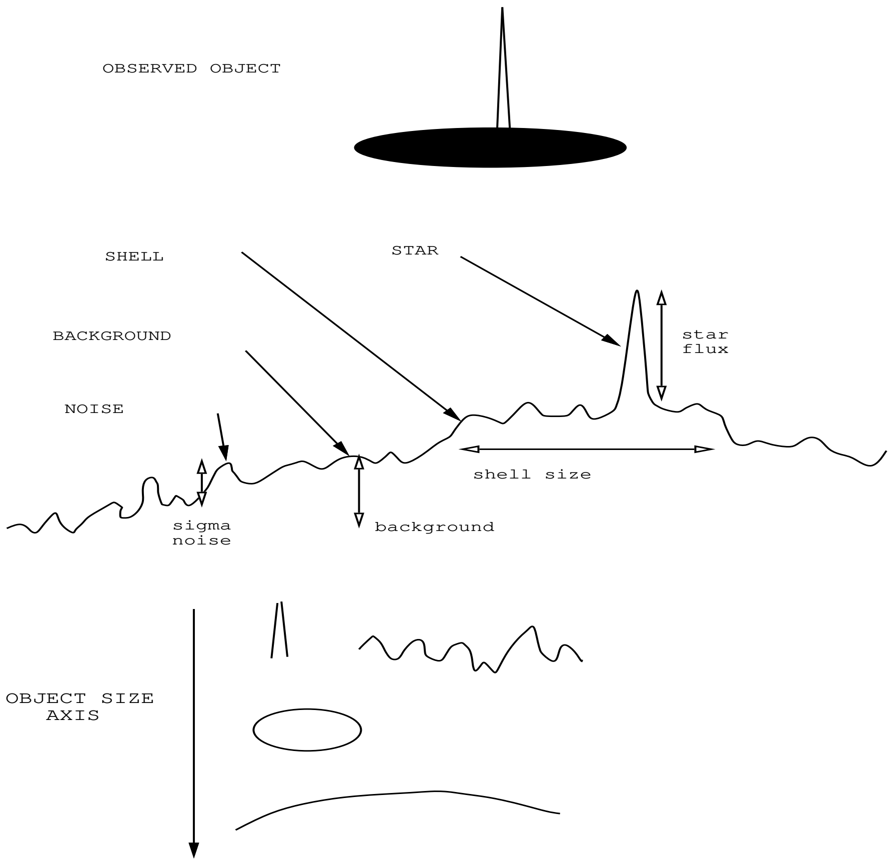
– Wavelet: Image Processing and Data Analysis, Starck et al, Cambridge University Press, 1998.

– Wavelet: Wavelets, a Mathematical Tools for Signal Analysis, C.K. Chui, SIAM, 1997.

– Deconvolution: Introduction to Inverse Problems in Imaging, Bertero and Boccaci, Institute of Physics Publishing, Bristol and Philadelphia, 1998.

– Image Processing: Fundamentals of Digital Image Processing, A.K. Jain, Prentice Hall, International Editions, 1989.

Why multiresolution ?



The Continuous Wavelet Transform

$$W(a,b) = K \int_{-\infty}^{+\infty} \psi^*(\frac{x-b}{a}) f(x) dx$$

where:

- $W(a,b)$ is the wavelet coefficient of the function $f(x)$
- $\psi(x)$ is the analyzing wavelet
- a (> 0) is the scale parameter
- b is the position parameter

In Fourier space, we have: $\hat{W}(a,\nu) = \sqrt{a}\hat{f}(\nu)\hat{\psi}^*(a\nu)$
When the scale a varies, the filter $\hat{\psi}^*(a\nu)$ is only reduced or dilated while keeping the same pattern.

Properties

- CWT is a linear transformation:
  - if $f(x) = f_1(x) + f_2(x)$ then $W_f(a,b) = W_{f_1}(a,b) + W_{f_2}(a,b)$
  - if $f(x) = kf_1(x))$ then $W_f(a,b) = kW_{f_1}(a,b)$
- CWT is covariant under translation:

$$\text{if } f_0(x) = f(x - x_0) \text{ then } W_{f_0}(a,b) = W_f(a, b - x_0)$$

- CWT is covariant under dilation:

$$\text{if } f_s(x) = f(sx) \text{ then } W_{f_s}(a,b) = \frac{1}{s^{\frac{1}{2}}} W_f(sa, sb)$$

Whatever the scale and the position, the signal analysis is done using the same function.

The Inverse Transform

The inverse transform is:

$$f(x) = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} \frac{1}{\sqrt{a}} W(a,b) \psi(\frac{x-b}{a}) \frac{da db}{a^2}$$

where

$$C_\psi = \int_{-\infty}^{+\infty} \mid \hat{\psi}(t) \mid^2 \frac{dt}{t} < +\infty$$

Reconstruction is only possible if C_ψ is defined (admissibility condition). This condition implies $\hat{\psi}(0) = 0$, i.e. the mean of the wavelet function is 0.

Morlet's Wavelet

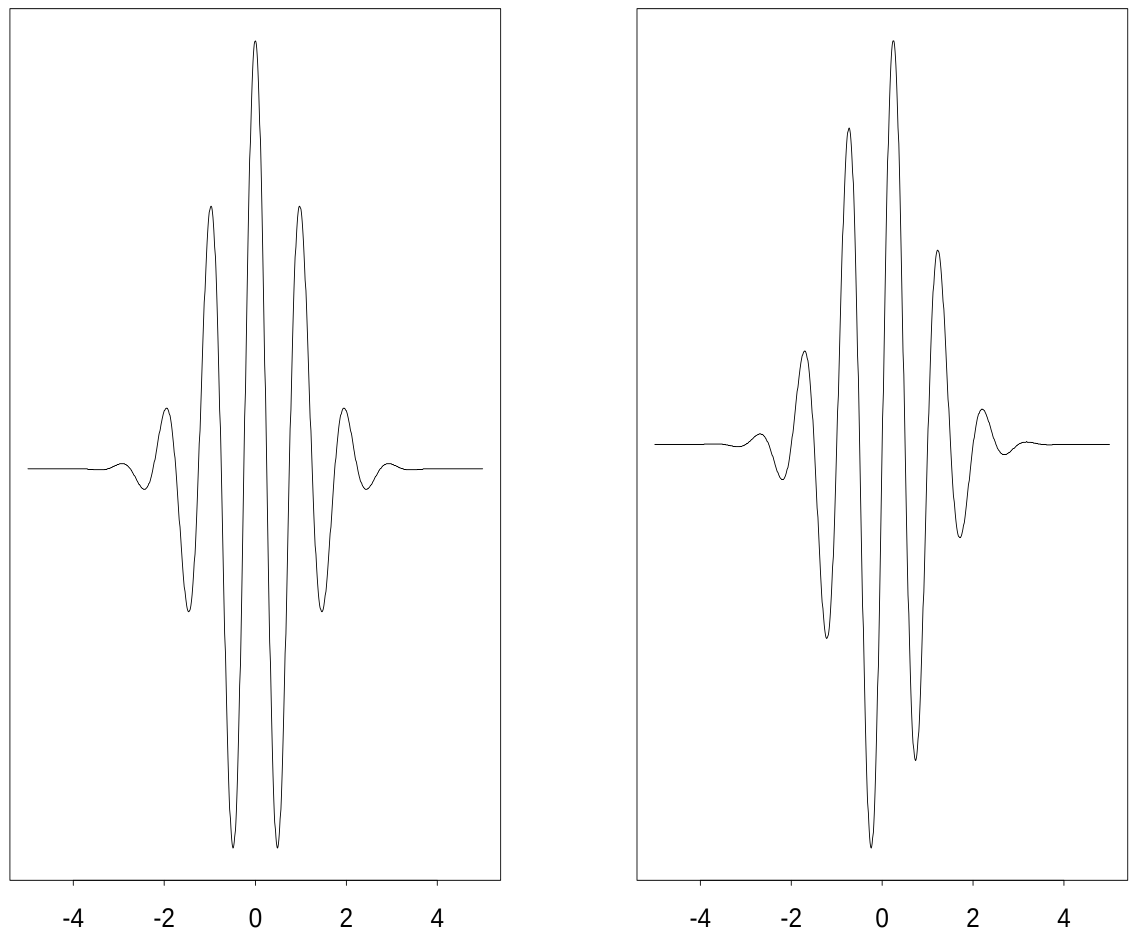
The wavelet defined by Morlet is:

$$\hat{g}(\nu) = e^{-2\pi^2(\nu-\nu_0)^2}$$

This is a complex wavelet which can be decomposed into two parts, one for the real part, and the other for the imaginary part:

$$\begin{aligned} g_r(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cos(2\pi\nu_0 x) \\ g_i(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sin(2\pi\nu_0 x) \end{aligned}$$

where ν_0 is a constant. Morlet's transform is not admissible. For ν_0 greater than approximately 0.8 the mean of the wavelet function is very small, so that approximate reconstruction is satisfactory.

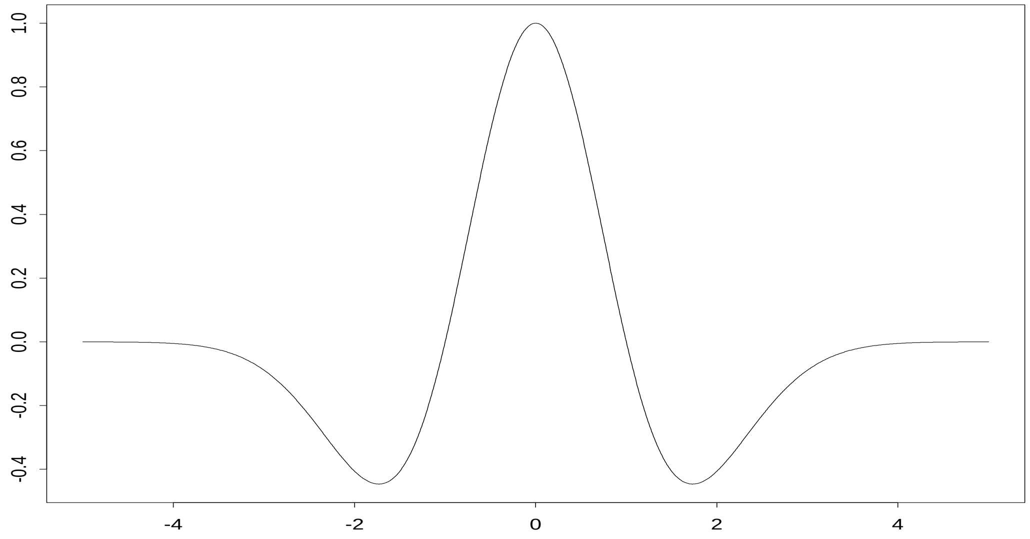

-4
-2
0
2
4
-4
-2
0
2
4

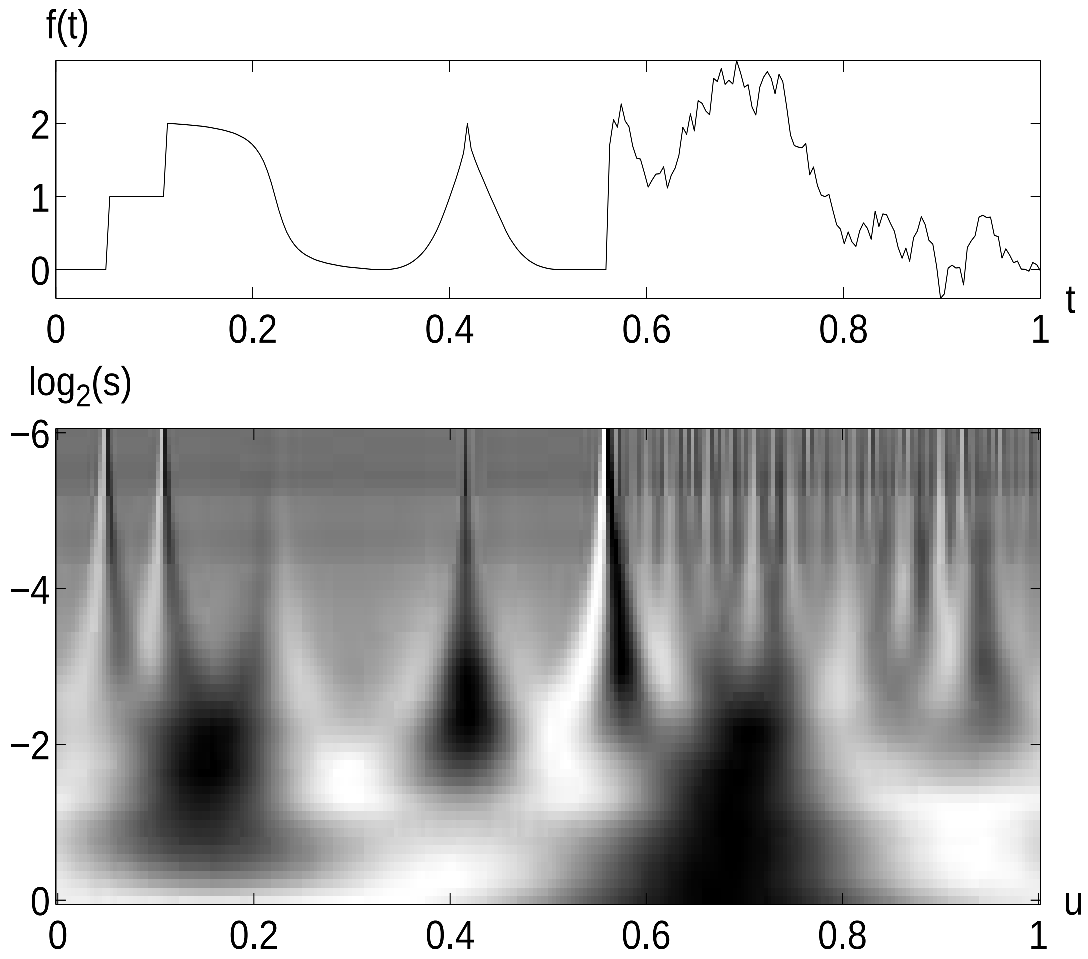
Mexican hat

The Mexican hat is in one dimension:

$$g(x) = (1 - x^2)e^{-\frac{1}{2}x^2}$$

This is the second derivative of a Gaussian.





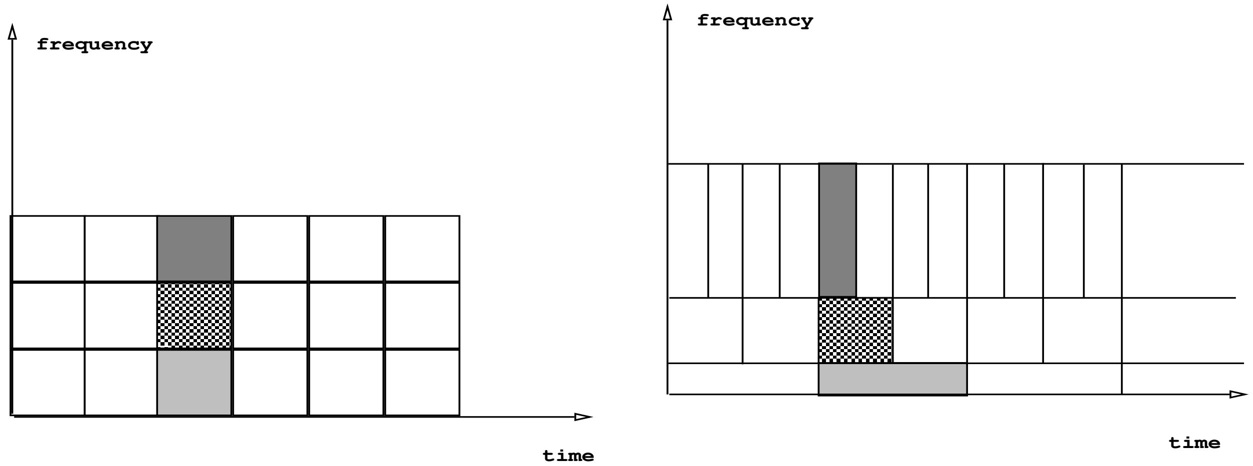
Black, grey, and white points correspond respectively to positive, zero and negative wavelet coefficients (figure from Mallat's book, 1998).

Comparison with the STFT

The Short Term Fourier Transform is defined by:

$$STFT(\nu, b) = \int_{\infty}^{+\infty} \exp(-j2\pi\nu t) f(t) g(t-b) dt$$

when g is a Gaussien, it corresponds to the Gabor transform.



Discrete Wavelet Transform

Several approaches:

1. Convolution
2. Mallat transform
3. Feauveau transform
4. à trous algorithm
5. Pyramidal algorithm
6. Pyramidal algorithm using the FFT

Multiresolution Analysis

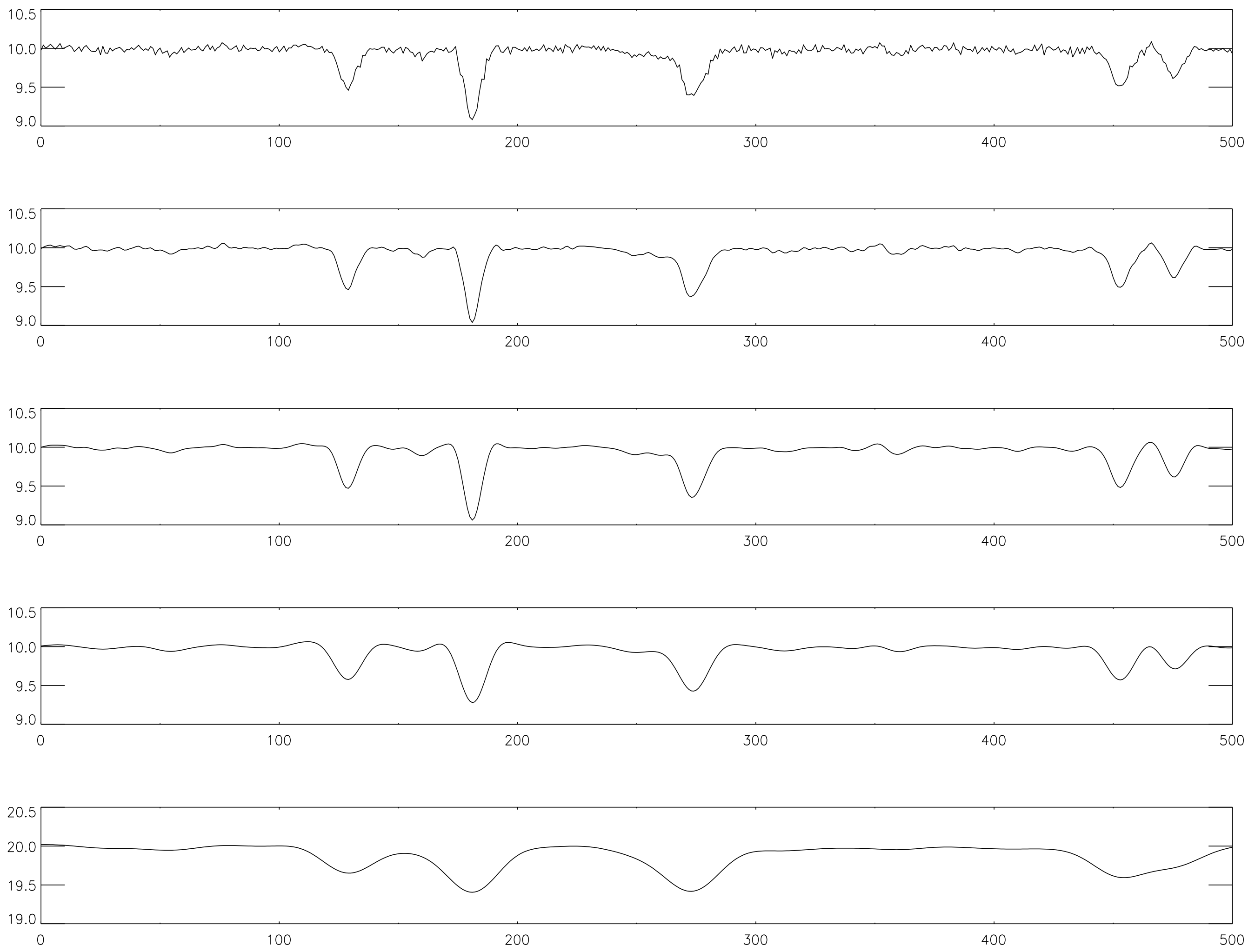
The multiresolution analysis (Mallat, 1989) results from the embedded subsets generated by the interpolations at different scales.

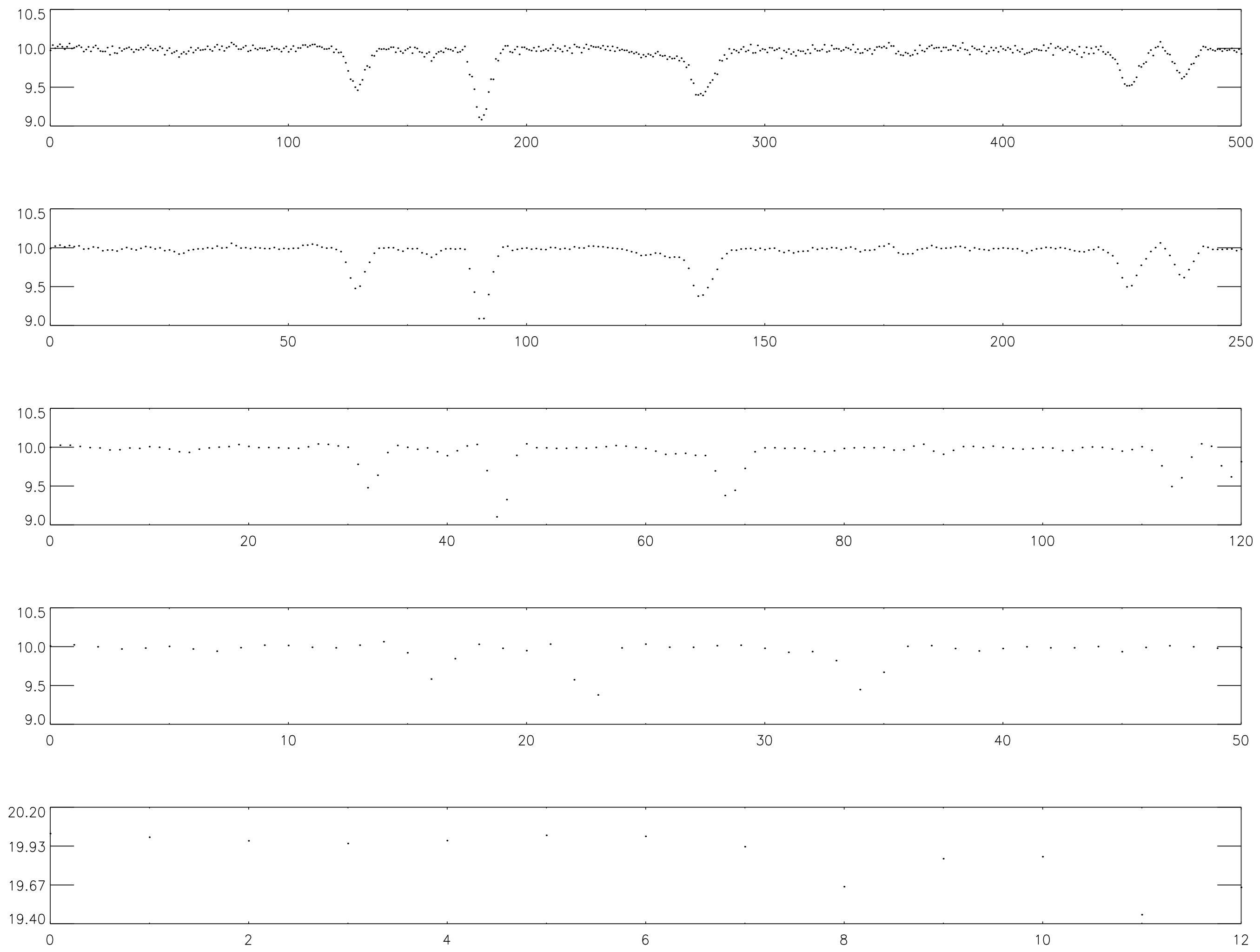
A function $f(x)$ is projected at each step j on the subset V_j $(\ldots\ldots \subset V_3 \subset V_2 \subset V_1 \subset V_0)$. This projection is defined by the scalar product $c_{j,k}$ of $f(x)$ with the scaling function $\phi(x)$ which is dilated and translated:

$$c_{j,k} = < f(x), \phi_{j,k}(x) >$$

$$\phi_{j,k}(x) = 2^{-j}\phi(2^{-j}x - k)$$

where $\phi(x)$ is the scaling function. ϕ is a low-pass filter.


10.5
10.0
9.5
9.0
0
100
200
300
400
500
20.5
20.0
19.5
19.0



10.5
10.0
9.5
9.0
0
100
200
300
400
500
10.5
10.0
9.5
9.0
0
50
100
150
200
250
10.5
10.0
9.5
9.0
0
20
40
60
80
100
120
10.5
10.0
9.5
9.0
0
10
20
30
40
50
20.20
19.93
19.67
19.40
0
2
4
6
8
10
12

Wavelets and Multiresolution Analysis

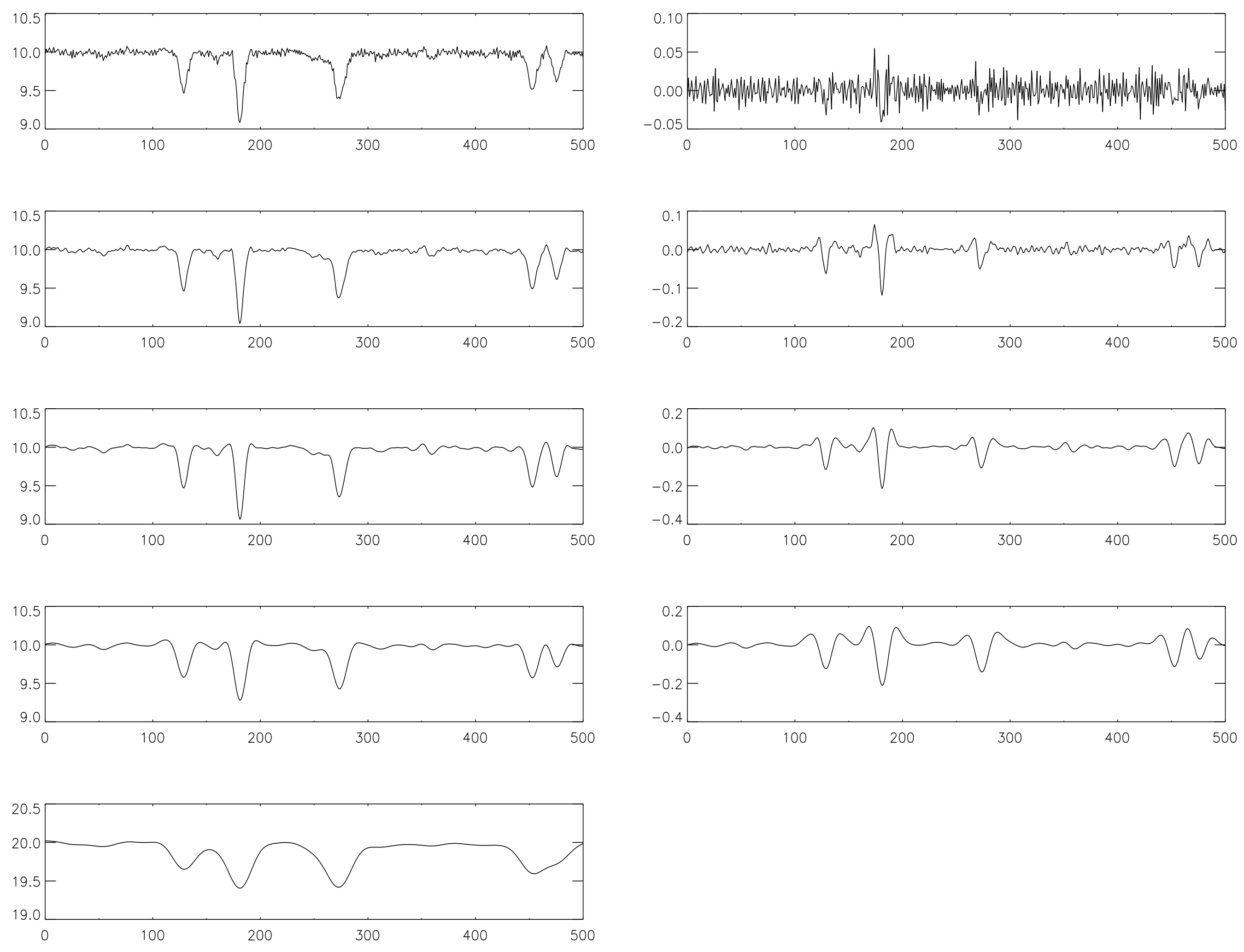
The difference between c_j and c_{j+1} is contained in the detail signal belonging to the space O_{j+1} orthogonal to V_{j+1}.

$$O_{j+1} \oplus V_{j+1} = V_j$$

The set $\{\sqrt{2^{-j}}\psi(2^{-j}x - k)\}_{k \in \mathcal{Z}}$ form a basis of O_j. $\psi(x)$ is the wavelet function.

the wavelet coefficients are obtained by:

$$w_{j,k} \quad = \quad < f(x), 2^{-j}\psi(2^{-j}x - k) >$$



the Fast Wavelet Transform

As $\phi(x)$ is a scaling function which has the property:
$\frac{1}{2}\phi(\frac{x}{2}) = \sum_n h(n)\phi(x-n)$. $c_{j+1,k}$ can be obtained by direct computation from $c_{j,k}$

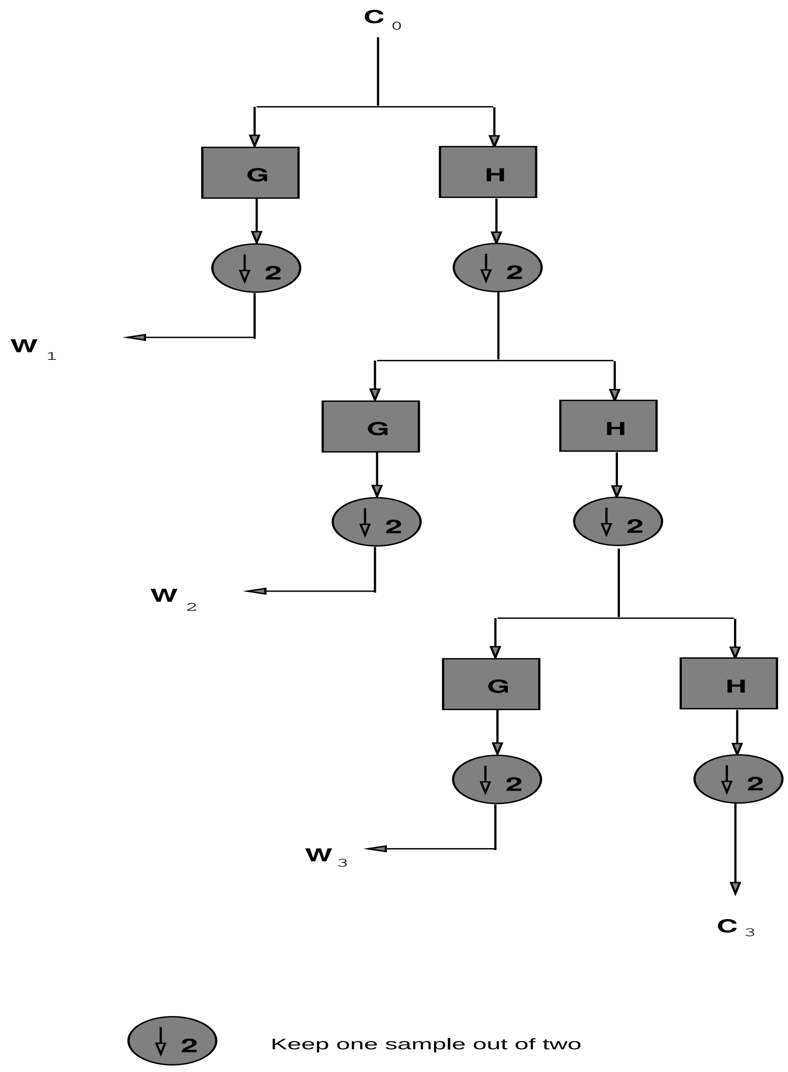
$$c_{j+1,k} = \sum_n h(n-2k)c_{j,n}$$

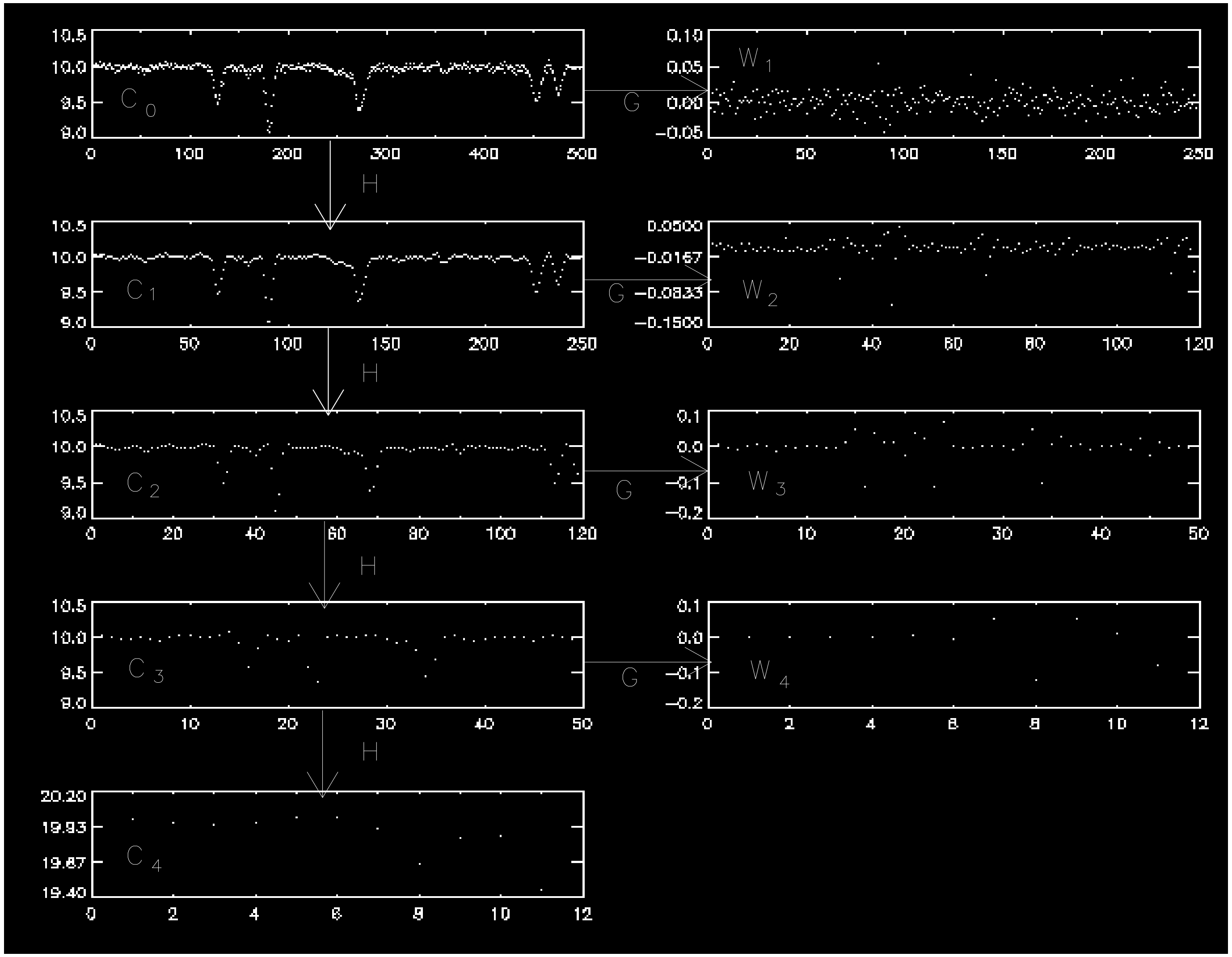
and $\frac{1}{2}\psi(\frac{x}{2}) = \sum_n g(n)\phi(x-n)$.
The scalar products $< f(x), 2^{-(j+1)}\psi(2^{-(j+1)}x-k) >$ are computed with:

$$w_{j+1,k} = \sum_n g(n-2k)c_{j,n}$$

Reconstuction by:

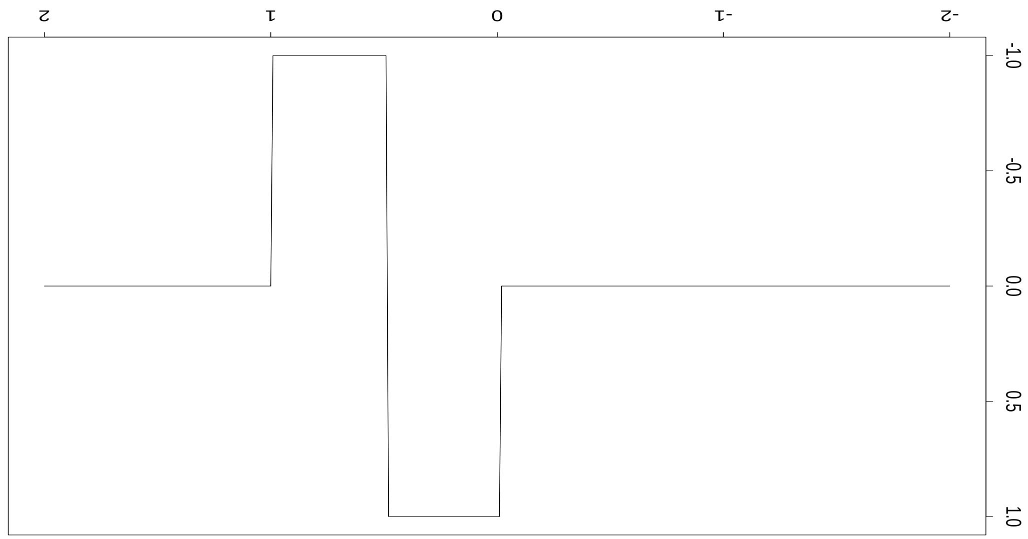
$$c_{j,k} = 2\sum_n h(k-2n)c_{j+1,n} + g(k-2n)w_{j+1,n}$$


C 0
G
H
↓ 2
W 1
W 2
W 3
C 3
↓ 2 Keep one sample out of two


C 0
C 1
C 2
C 3
C 4
W 1
W 2
W 3
W 4
H
G

Example: the Haar Wavelet Transform

$$\begin{aligned} \psi(x) &= 1 && \text{if } 0 \leq x < \tfrac{1}{2} \\ \psi(x) &= -1 && \text{if } \tfrac{1}{2} \leq x < 1 \\ \psi(x) &= 0 && \text{otherwise} \end{aligned}$$



64	48	16	32	56	56	48	24

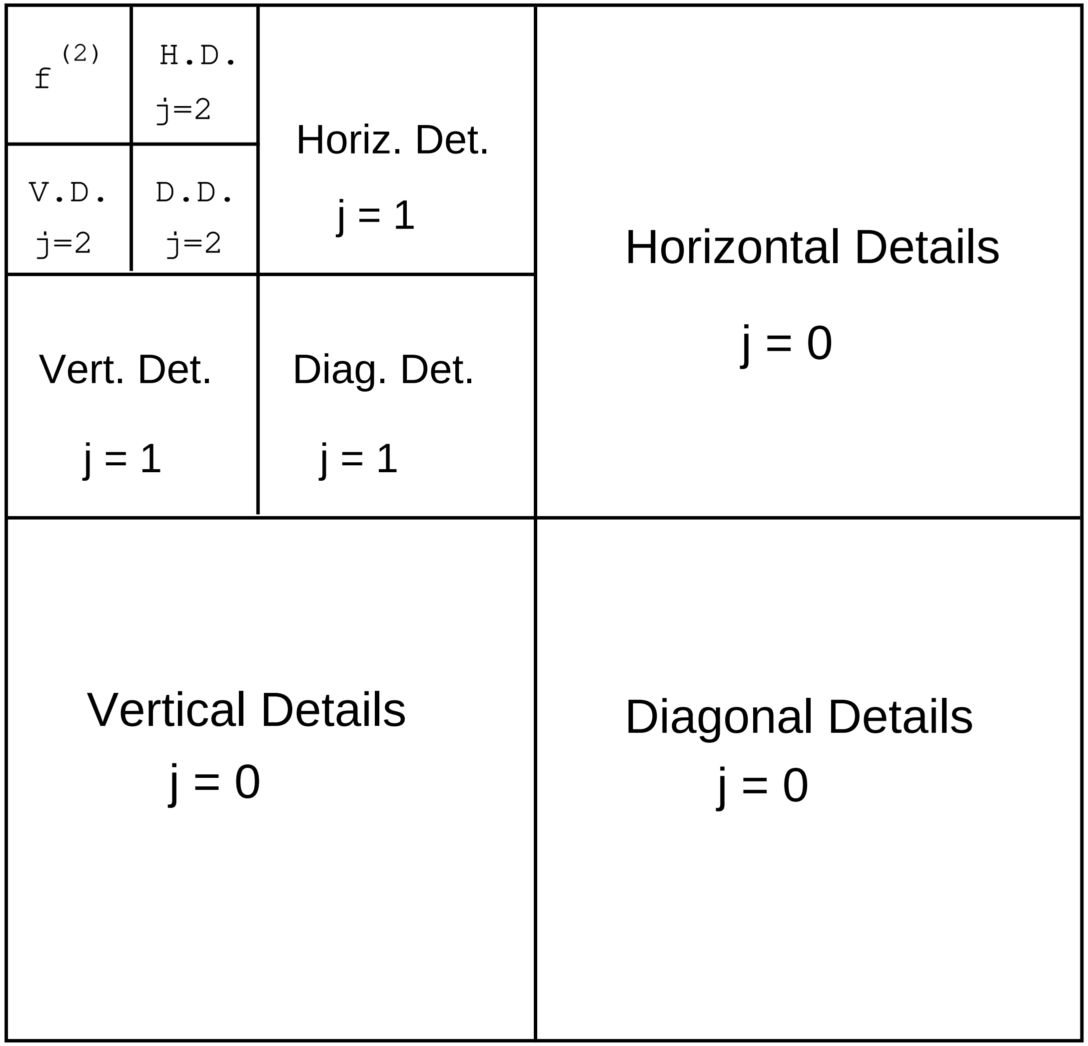
56	24	56	36	**8**	**-8**	**0**	**12**
40	46	**16**	**10**	**8**	**-8**	**0**	**12**
43	**-3**	**16**	**10**	**8**	**-8**	**0**	**12**

At two dimensions, we separate the variables x,y:

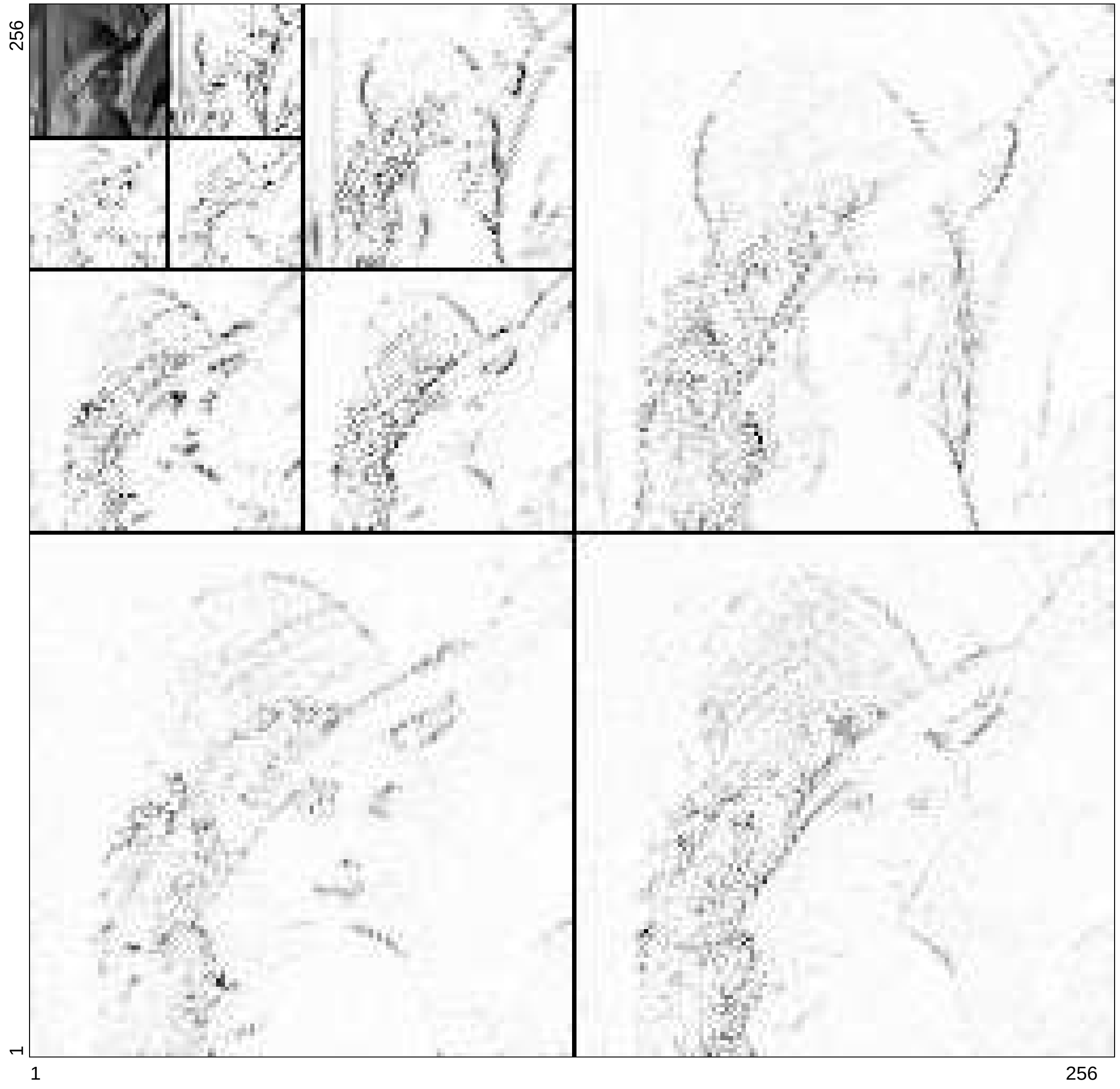
- vertical wavelet: $\psi^1(x,y) = \phi(x)\psi(y)$
- horizontal wavelet: $\psi^2(x,y) = \psi(x)\phi(y)$
- diagonal wavelet: $\psi^3(x,y) = \psi(x)\psi(y)$

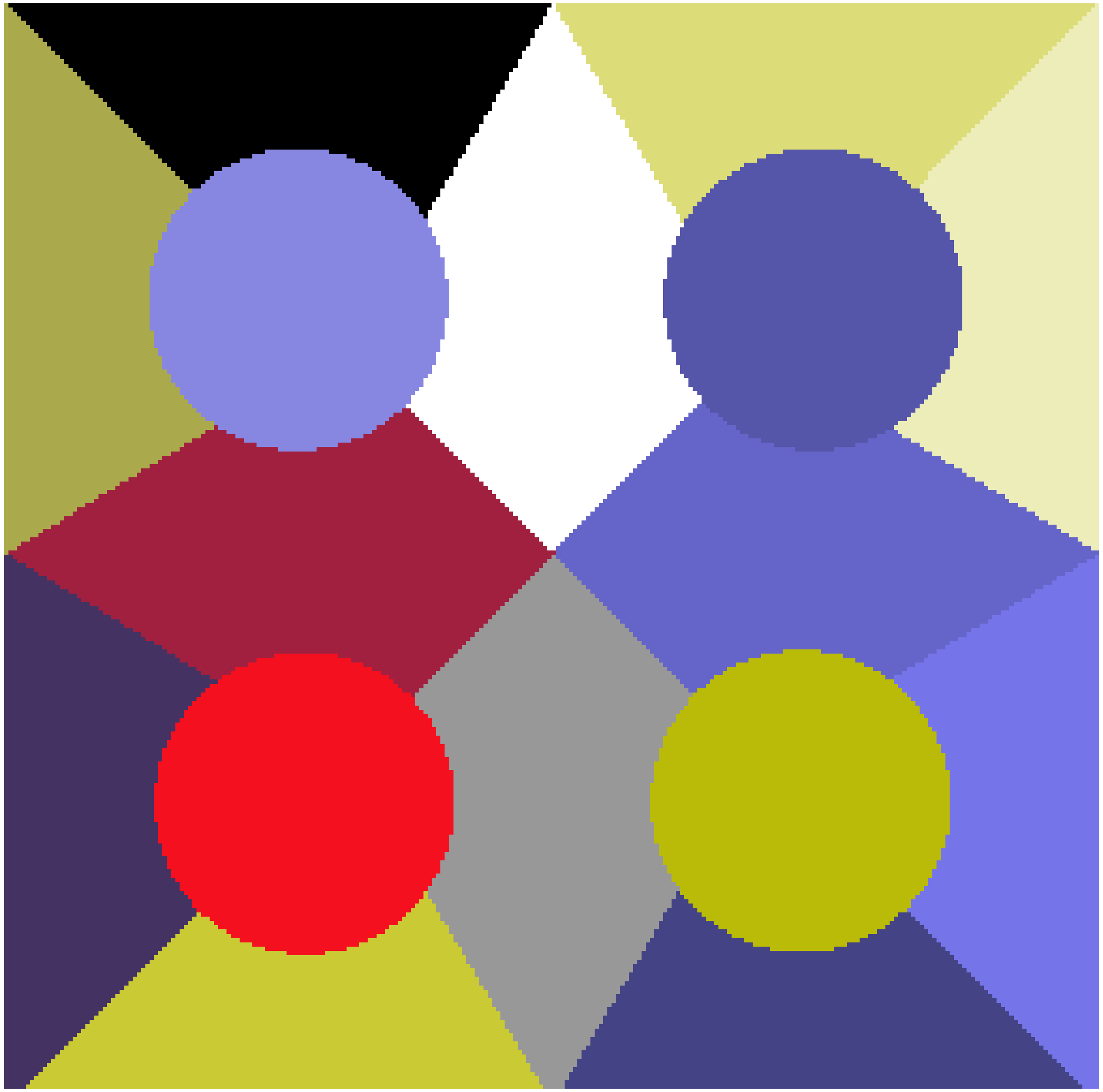
The detail signal is contained in three sub-images

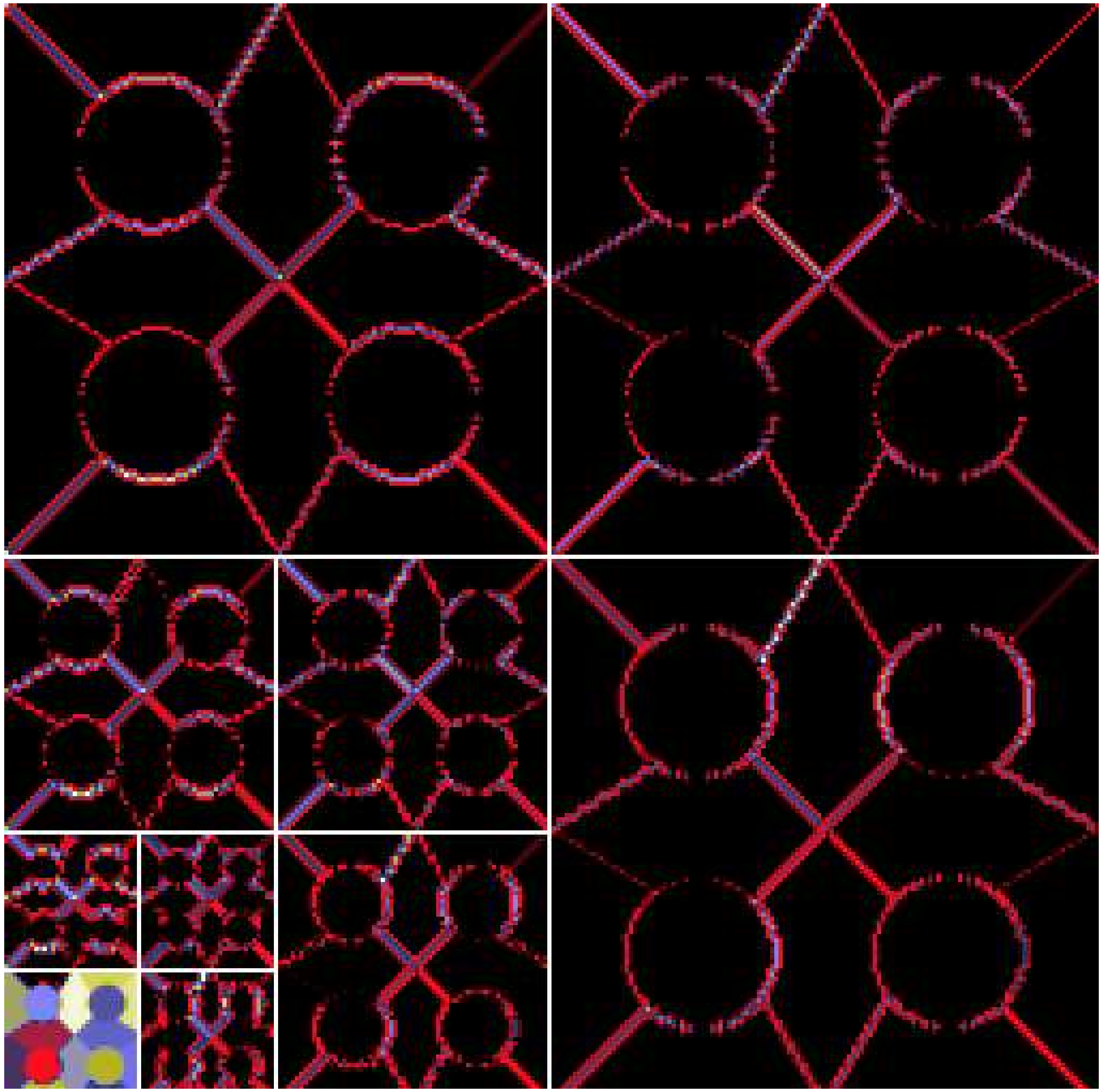
$$
\begin{aligned}
w_j^1(k_x, k_y) &= \sum_{l_x=-\infty}^{+\infty} \sum_{l_y=-\infty}^{+\infty} g(l_x - 2k_x) h(l_y - 2k_y) c_{j+1}(l_x, l_y) \\
w_j^2(k_x, k_y) &= \sum_{l_x=-\infty}^{+\infty} \sum_{l_y=-\infty}^{+\infty} h(l_x - 2k_x) g(l_y - 2k_y) c_{j+1}(l_x, l_y) \\
w_j^3(k_x, k_y) &= \sum_{l_x=-\infty}^{+\infty} \sum_{l_y=-\infty}^{+\infty} g(l_x - 2k_x) g(l_y - 2k_y) c_{j+1}(l_x, l_y)
\end{aligned}
$$


f (2)
H.D.
j=2
V.D.
j=2
D.D.
j=2
Horiz. Det.
j = 1
Vert. Det.
j = 1
Diag. Det.
j = 1
Horizontal Details
j = 0
Vertical Details
j = 0
Diagonal Details
j = 0


1
512
1
512


256
1
1
256





bi-orthogonal wavelets

It is a generalization of the orthogonal wavelets. Two other spaces $\tilde{O}_j$ and $\tilde{V}_j$ are introduced for the reconstruction:

- $V_{j-1} = V_j \oplus O_j$, and $V_j \not\perp O_j$
- $\tilde{V}_{j-1} = \tilde{V}_j \oplus \tilde{O}_j$, and $\tilde{V}_j \not\perp \tilde{O}_j$
- $\tilde{O}_j \perp V_j$ and $O_j \perp \tilde{V}_j$

Reconstruction with bi-orthogonal wavelets

Using two other filters $\tilde{h}$ and $\tilde{g}$, defined to be conjugate to h and g. The reconstruction of the signal is performed with:

$$c_{j,k} = 2 \sum_l [c_{j+1,l} \tilde{h}(k - 2l) + w_{j+1,l} \tilde{g}(k - 2l)]$$

In order to get an exact reconstruction, two conditions are required for the conjugate filters:

- *Dealiasing condition*: $\hat{h}(\nu + \frac{1}{2})\hat{\tilde{h}}(\nu) + \hat{g}(\nu + \frac{1}{2})\hat{\tilde{g}}(\nu) = 0$
- *Exact restoration:* $\hat{h}(\nu)\hat{\tilde{h}}(\nu) + \hat{g}(\nu)\hat{\tilde{g}}(\nu) = 1$

Orthogonal wavelets

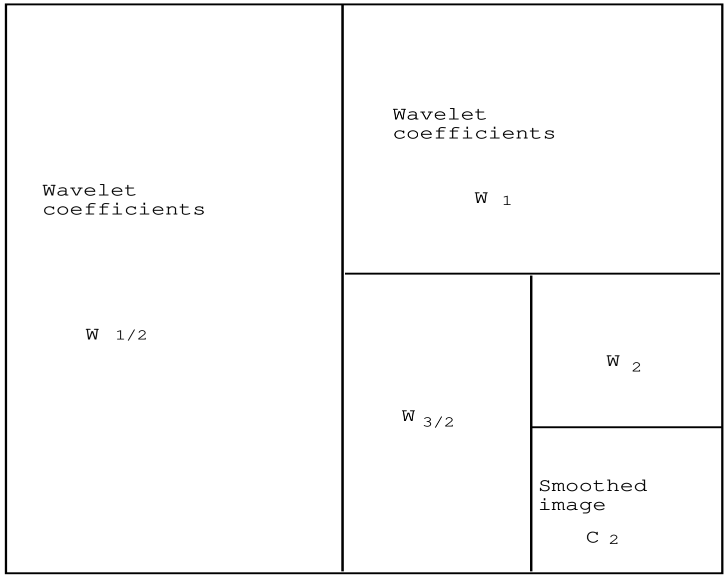
Orthogonal wavelets correspond to the restricted case where:

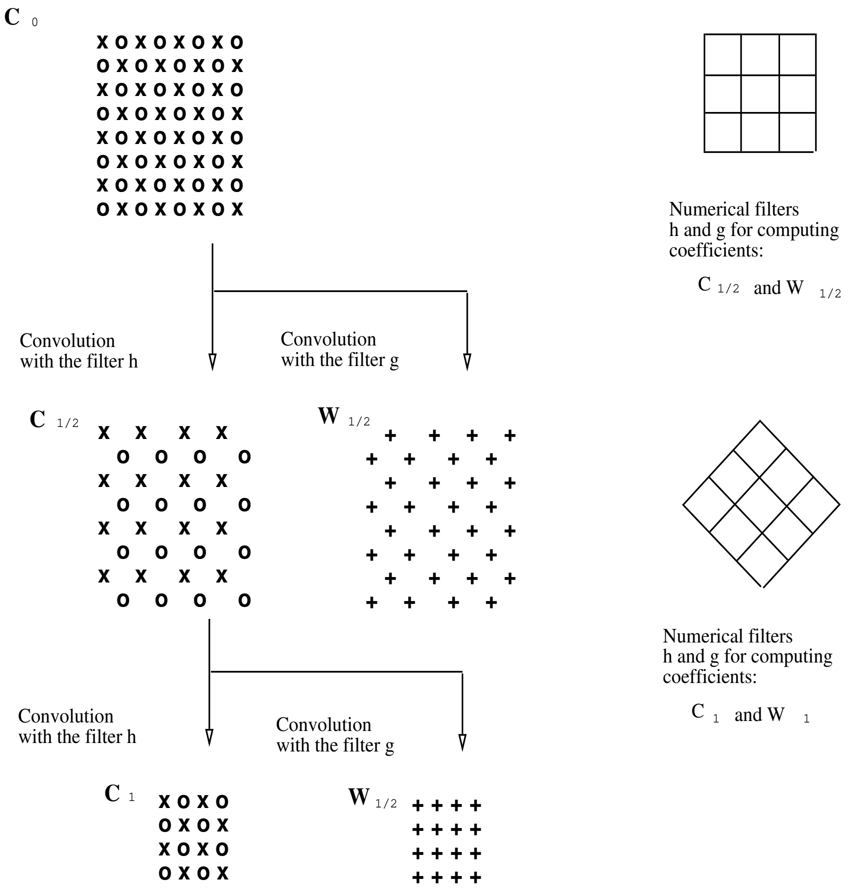
$$\begin{aligned}\hat{g}(\nu) &= e^{-2\pi i\nu}\hat{h}^*(\nu+\frac{1}{2}) \\ \hat{\tilde{h}}(\nu) &= \hat{h}^*(\nu) \\ \hat{\tilde{g}}(\nu) &= \hat{g}^*(\nu)\end{aligned}$$

and $\mid \hat{h}(\nu) \mid^2 + \mid \hat{h}(\nu+\frac{1}{2}) \mid^2 = 1$
Then we have $g(n) = (-1)^{1-n}h(1-n)$.
$g(n)$ and $h(n)$ are Conjugate Mirror Filters.

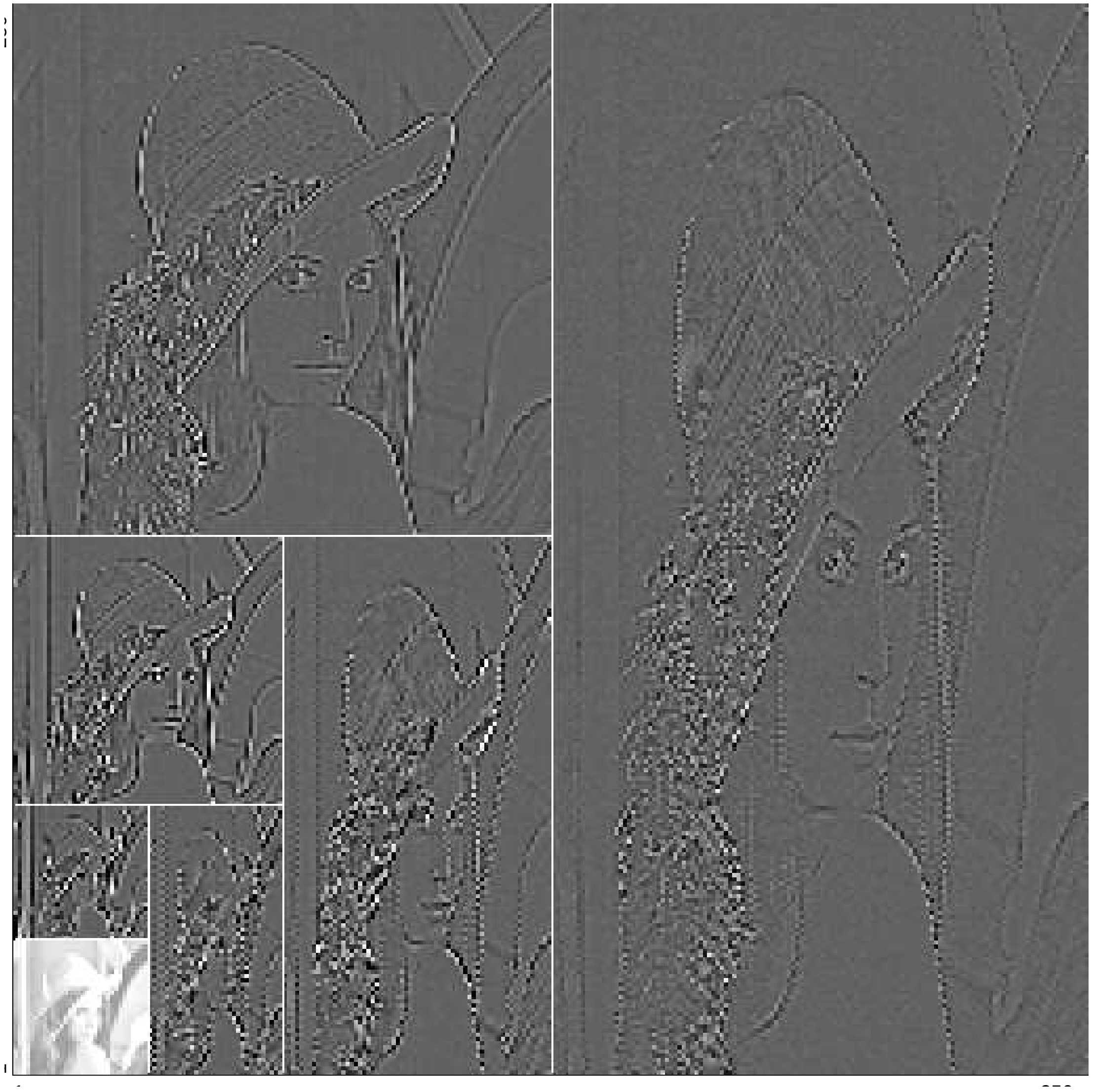
Non-Dyadic Resolution Factor

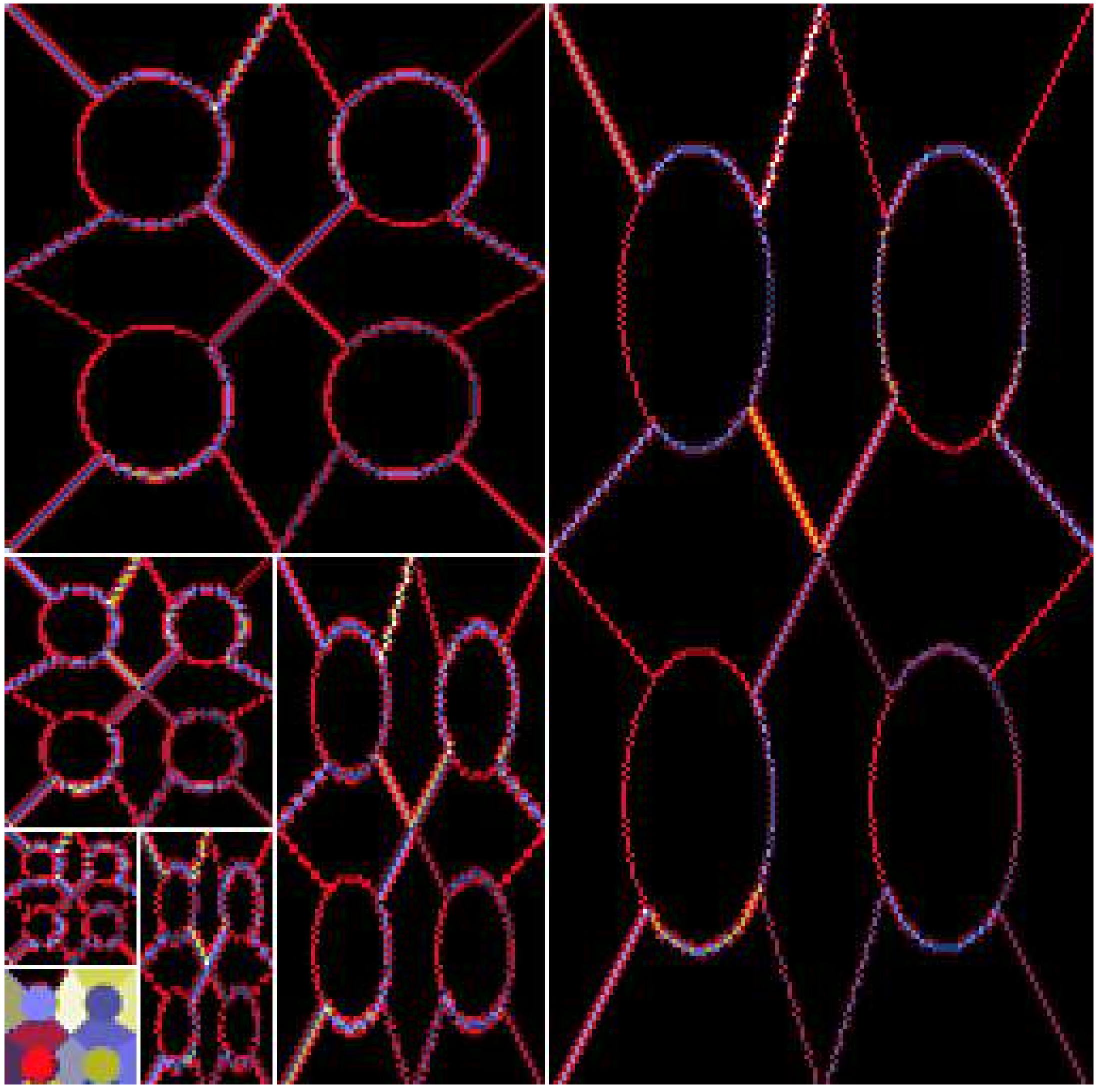
Feauveau introduced *quincunx* analysis. This analysis is not dyadic and allows an image decomposition with a resolution factor equal to $\sqrt{2}$.



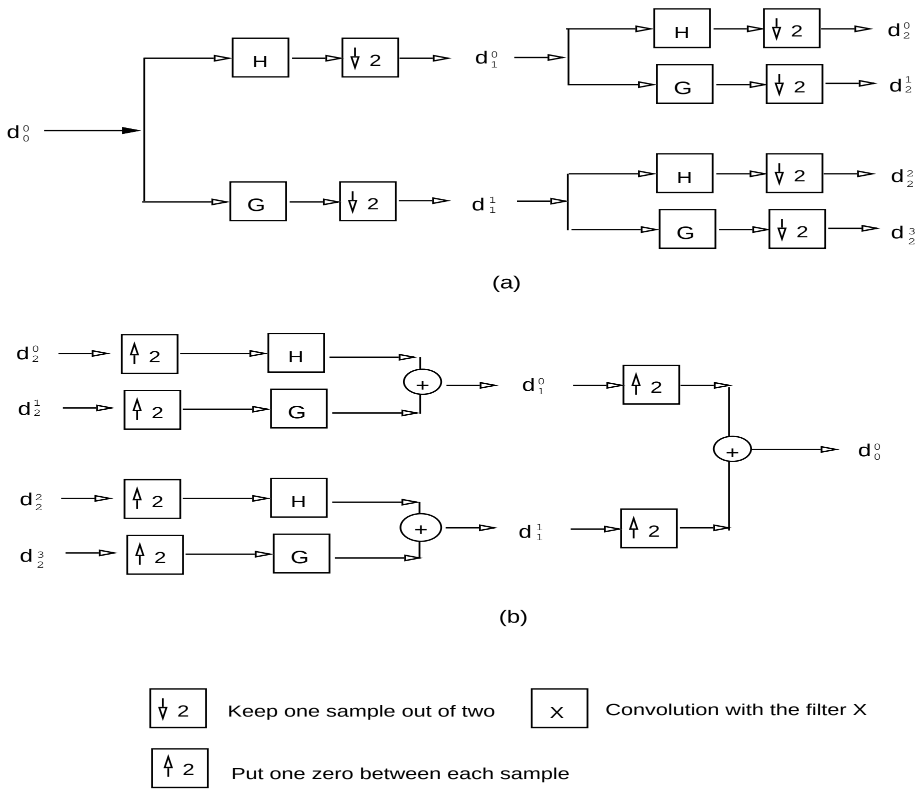


At each step, the image is undersampled by two in one direction (x and y, alternatively). This undersampling is made by keeping one pixel out of two, alternatively even and odd.





Wavelet Packet



Wavelet packets were introduced by Coifman, Meyer and Wickerhayser (1992). Instead of dividing only the approximation space, as in the standard orthogonal wavelet transform, detail spaces are also divided. *(a) forward direction, (b) reconstruction.*